

# Correlation functions with fusion-channel multiplicity in $\mathcal{W}_3$ Toda field theory

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**ABSTRACT:** Current studies of  $\mathcal{W}_N$  Toda field theory focus on correlation functions such that the  $\mathcal{W}_N$  highest-weight representations in the fusion channels are multiplicity-free. In this work, we study  $\mathcal{W}_3$  Toda 4-point functions with multiplicity in the fusion channel. The conformal blocks of these 4-point functions involve matrix elements of a fully-degenerate primary field with a highest-weight in the *adjoint* representation of  $\mathfrak{sl}_3$ , and a fully-degenerate primary field with a highest-weight in the fundamental representation of  $\mathfrak{sl}_3$ . We show that, when the fusion rules do not involve multiplicities, the matrix elements of the fully-degenerate adjoint field, between two arbitrary descendant states, can be computed explicitly, on equal footing with the matrix elements of the semi-degenerate fundamental field. Using null-state conditions, we obtain a fourth-order Fuchsian differential equation for the conformal blocks. Using Okubo theory, we show that, due to the presence of multiplicities, this differential equation belongs to a class of Fuchsian equations that is different from those that have appeared so far in  $\mathcal{W}_N$  theories. We solve this equation, compute its monodromy group, and construct the monodromy-invariant correlation functions. This computation shows in detail how the ambiguities that are caused by the presence of multiplicities are fixed by requiring monodromy-invariance.

**KEYWORDS:**  $\mathcal{W}_N$  algebra, 2-dimensional conformal field theory, Fuchsian differential equations.

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## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b><math>\mathcal{W}_3</math> conformal field theory</b>	<b>3</b>
2.1	$\mathcal{W}_3$ chiral symmetry algebra	3
2.2	Highest-weight modules	6
2.3	The Coulomb gas approach	9
<b>3</b>	<b>Conformal blocks and matrix elements</b>	<b>11</b>
3.1	Power series expansion of the conformal block	11
3.2	The Shapovalov matrix of inner products	12
3.3	The matrix elements	12
<b>4</b>	<b>A fourth-order Fuchsian differential equation</b>	<b>17</b>
4.1	Okubo theory: classification of Fuchsian systems	21
<b>5</b>	<b>Constructing monodromy-invariant function</b>	<b>24</b>
5.1	The monodromy group	27
5.2	Monodromy-invariant correlation function	28
<b>6</b>	<b>Summary and discussion</b>	<b>31</b>
<b>A</b>	<b>Shapovalov matrices, at level-one and level-two</b>	<b>32</b>
<b>B</b>	<b>Matrix elements, at level-one and at level-two</b>	<b>33</b>
<b>C</b>	<b><math>\mathcal{W}_3</math> highest-weight modules with null-states at level-two</b>	<b>35</b>
<b>D</b>	<b>Structure constants from the <math>\mathfrak{sl}_3</math> Coulomb gas</b>	<b>36</b>
D.1	The first node	36
D.2	The second node	37
<b>E</b>	<b>A <math>\mathcal{W}_3</math> basis</b>	<b>38</b>
E.1	Definitions	38
E.2	Expanding a disordered state in terms of basis states	40

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# 1 Introduction

A 2-dimensional conformal field theory is based on the representation theory of an infinite-dimensional algebra. In the absence of an extended symmetry, the infinite-dimensional algebra is the Virasoro algebra generated by the spin-2 energy-momentum tensor  $\mathcal{T}(z)$ . The most direct generalisations of the Virasoro algebra are the  $\mathcal{W}_N$  algebras,  $N = 2, 3, \dots$ , generated by  $(N-1)$  holomorphic fields of spin-2,  $\dots$ , spin- $N$  [1]. For a review, see [2]. The main goal in any conformal field theory is to compute the correlation functions of the primary fields. One way to achieve this is based on the fact that there are null-states in the highest-weight representations that correspond to the primary fields [3]. This approach allows one to obtain differential equations for the conformal blocks which are the building blocks of the correlation function. However, for  $\mathcal{W}_N$  models, and particularly for primary fields in higher representations of  $\mathfrak{sl}_N$ , this program contains a number of additional subtleties.

In principle, using operator product expansions, any conformal block can be constructed as a series expansion in its holomorphic coordinates. To this end, one needs to compute matrix elements of the primary fields between two arbitrary descendant states. In  $\mathcal{W}_N$  theories, as opposed to Virasoro theories, multiplicities can appear in the tensor products of the irreducible highest-weight representations. Consequently, the  $\mathcal{W}_N$  Ward identities are not sufficient to express the 3-point functions that involve descendant states in terms of 3-point functions that involve primary states only. This has, so far, limited the applicability of this approach to correlation functions in which all fields, except two, are *semi-degenerate*, with (at least) one level-one null-state [4, 5]. These semi-degenerate fields belong to the *fundamental* representation of  $\mathfrak{sl}_N$ . In [6], Kanno *et. al* showed that all 3-point matrix elements that contain one semi-degenerate fundamental field can be expressed in terms of 3-point functions of primary fields.

Aside from the above approach, the  $\mathfrak{sl}_N$  Coulomb gas approach [1, 4, 5, 7] provides a representation of  $\mathcal{W}_N$  correlation functions in terms of multi-dimensional integrals, provided that a charge-neutrality condition is obeyed. The Coulomb gas correlation functions contain primary fields in general  $\mathfrak{sl}_N$  representations. However, the evaluation of these multi-dimensional integrals is in general possible only in the absence of multiplicities.

To the best of our knowledge, all conformal blocks, currently available in the literature *in explicit form*, are such that the multiplicities are not present in the fusion channel [8–11]. The purpose of this paper is to provide explicit results for correlation functions with such multiplicities. We focus on  $\mathcal{W}_3$  Toda theory, and consider 4-point correlation functions that admit highest-weight representations with multiplicities in the fusion-channel. These 4-point functions involve a fully-degenerate primary field with a charge-vector in the *adjoint* representation of  $\mathfrak{sl}_3$ , and a fully-degenerate primary field with a charge-vector in the *fundamental* representation of  $\mathfrak{sl}_3$ . We show that, when the fusion rules are obeyed and do not involve multiplicities, all matrix elements of the fully-degenerate adjoint field, between two arbitrary descendant states, can be explicitly computed in terms of matrix

elements of primary fields. For completeness, we provide a parallel discussion of the explicit computation of all matrix elements of the semi-degenerate fundamental field, following [6]. Given these matrix elements, we obtain explicit series expansion for the conformal blocks.

Next, we show that the conformal blocks formed from the above matrix elements satisfy a fourth-order Fuchsian differential equation. We verify that, with the appropriate choice of the parameters, the series expansion of the conformal block, obtained matrix elements, agrees with the differential equation.

The remaining part of the paper is devoted to constructing the modular-invariant correlation function, that corresponds to the previously-computed matrix elements and conformal blocks, on the basis of the analysis of the differential equation. Using Okubo's matrix form [12] of the Fuchsian differential equations, we find that, notwithstanding the presence of an integral difference among the characteristic exponents, this equation is free from accessory parameters and belongs to type-II in Okubo's classification. Because the fusion rules are obeyed, the conformal blocks can be written in terms of Coulomb gas integrals. Using the Coulomb gas integral representation, we compute explicitly the monodromy group of the differential equation, and show that requiring monodromy invariance resolves an ambiguity that arises in the matrix elements when states from a highest-weight representation with multiplicity are involved.

The paper is organised as follows. In Section 2, we introduce our notation and recall basic facts regarding  $\mathcal{W}_3$  conformal field theory. In 3, we compute the matrix elements of the fully-degenerate adjoint field between two arbitrary descendant states, and discuss the construction of the conformal blocks. In 4, we focus on a specific 4-point correlation function in  $\mathcal{W}_3$  theory, and derive a fourth-order Fuchsian differential equation that the conformal blocks, that are the building blocks of the correlation function, satisfy. In 5, we discuss construction of the monodromy-invariant correlation functions using the differential equation found in 4. In 6, we present our conclusion and discuss open problems. In appendix A, we discuss the Shapovalov matrix elements that we need in computations. In B, we compute matrix elements. In C, we discuss  $\mathcal{W}_3$  highest-weight modules with null-states at level-2. In E, we outline an algorithm to expand any  $\mathcal{W}_3$  state in terms of basis states.

## 2 $\mathcal{W}_3$ conformal field theory

### 2.1 $\mathcal{W}_3$ chiral symmetry algebra

$\mathcal{W}_3$  is an associative algebra generated by the modes  $L_n$  and  $W_n$  of the spin-2 energy-momentum tensor  $\mathcal{T}(z)$  and of the spin-3 holomorphic field  $\mathcal{W}(z)$ . These modes are defined by their action on a field  $\Phi(w)$  that corresponds to a state in  $\mathcal{W}_3$  theory,

$$L_n \Phi(w) = \frac{1}{2\pi i} \oint_{\mathcal{C}_w} dz (z-w)^{n+1} \mathcal{T}(z) \Phi(w), \quad (2.1)$$

$$W_n \Phi(w) = \frac{1}{2\pi i} \oint_{\mathcal{C}_w} dz (z-w)^{n+2} \mathcal{W}(z) \Phi(w). \quad (2.2)$$

The Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \quad (2.3)$$

appears as a subalgebra of the  $\mathcal{W}_3$  one. The parametrisation of the  $\mathcal{W}_3$  central charge  $c$ , commonly used in the Toda field theory literature, is

$$c = 2 + 24 Q^2, \quad Q = b + \frac{1}{b}, \quad (2.4)$$

where  $Q$  is proportional to the background charge. The full  $\mathcal{W}_3$  algebra is given by the following two commutation relations,

$$[L_m, W_n] = (2m-n)W_{m+n}, \quad (2.5)$$

$$\begin{aligned} [W_m, W_n] = & \frac{1}{3}(m-n)\Lambda_{m+n} \\ & + \left( \frac{22+5c}{48} \right) \left( \frac{m-n}{30} \right) (2m^2 - mn + 2n^2 - 8) L_{m+n} \\ & + \left( \frac{22+5c}{48} \right) \left( \frac{c}{3 \cdot 5!} \right) (m^2 - 4)(m^3 - m) \delta_{m+n,0}, \end{aligned} \quad (2.6)$$

where  $\Lambda_m$  are the modes of the quasi-primary field  $\Lambda =: \mathcal{T}^2 : - \frac{3}{10} \partial^2 \mathcal{T}$ , the colons  $:$  stand for normal-ordering, and

$$\Lambda_m = \sum_{p \leq -2} L_p L_{m-p} + \sum_{p \geq -1} L_{m-p} L_p - \frac{3}{10}(m+2)(m+3)L_m. \quad (2.7)$$

In (2.6) we are assuming the following normalisation for the  $\mathcal{W}_3$  2-point function:

$$\langle \mathcal{W}(1) \mathcal{W}(0) \rangle = \eta \equiv \left( \frac{22+5c}{48} \right). \quad (2.8)$$

**$\mathcal{W}_3$  highest-weight modules.** A  $\mathcal{W}_3$  primary field  $\Phi_{h,q}(z)$  corresponds to a state  $|h, q\rangle$  which is labelled by the eigenvalues,  $h$  and  $q$ , of  $L_0$  and  $W_0$ ,

$$L_0 |h, q\rangle = h |h, q\rangle, \quad W_0 |h, q\rangle = q |h, q\rangle. \quad (2.9)$$

and which is annihilated by all positive modes of  $\mathcal{T}(z)$  and  $\mathcal{W}(z)$ ,

$$\forall n > 0, \quad L_n |h, q\rangle = 0, \quad W_n |h, q\rangle = 0 \quad (2.10)$$

A  $\mathcal{W}_3$  highest-weight representation is spanned by the basis states,

$$\mathcal{L}_I |h, q\rangle \equiv L_{-i_m} \cdots L_{-i_1} W_{-j_n} \cdots W_{-j_1} |h, q\rangle, \quad (2.11)$$

where the sets of positive integers  $I = \{i_m, \dots, i_1; j_n, \dots, j_1\}$ , are normal-ordered,

$$i_m \geq \dots \geq i_1 \geq 1, \quad j_n \geq \dots \geq j_1 \geq 1. \quad (2.12)$$

We use  $\{\emptyset; j_n, \dots, j_1\}$  and  $\{i_n, \dots, i_1; \emptyset\}$  when no  $L_i$ , and  $W_i$  modes act on the state, respectively, and  $I = \{\emptyset, \emptyset\}$ , when neither type of modes act on the state,  $\mathcal{L}_\emptyset |h, q\rangle = |h, q\rangle$ . In the following we will use the notation  $\Phi^{(I)X} = \mathcal{L}_I \Phi_X$  to denote a descendant field associated to the primary field  $\Phi_X$ , where  $X$  indexes the quantum numbers  $h, q$ . For instance  $\Phi^{\{2,1;3,1,1\}1} = L_{-2} L_{-1} W_{-3} W_{-1}^2 \Phi_{h_1, q_1}$ . That any  $\mathcal{W}_3$  highest-weight representation is spanned by the states (2.11), is typically stated without proof in the literature. For completeness, we outline a proof in [E](#).

**$\mathcal{W}_3$  Ward identities.** To construct the correlation functions of the primary fields we will need the conformal Ward identities. For the correlation function with additional insertion of the stress-energy tensor  $\mathcal{T}(z)$  one finds

$$\left\langle \mathcal{T}(z) \prod_{i=1}^N \Phi_i(z_i) \right\rangle = \sum_{i=1}^N \left( \frac{\Delta_i}{(z - z_i)^2} + \frac{1}{z - z_i} \partial_{z_i} \right) \left\langle \prod_{i=1}^N \Phi_i(z_i) \right\rangle, \quad (2.13)$$

where  $\Phi_i \equiv \Phi_{h_i, q_i}$ . From the asymptotic behavior of  $\mathcal{T}(z)$ ,  $\lim_{z \rightarrow \infty} \mathcal{T}(z) \sim 1/z^4$ , Equation (2.13) implies the following three identities,

$$\sum_{j=0}^N \partial_{z_j} \left\langle \prod_{i=1}^N \Phi_i(z_i) \right\rangle = 0, \quad (2.14)$$

$$\sum_{j=0}^N \left( z_j \partial_{z_j} + \Delta_j \right) \left\langle \prod_{i=1}^N \Phi_i(z_i) \right\rangle = 0, \quad (2.15)$$

$$\sum_{j=0}^N \left( z_j^2 \partial_{z_j} + 2z_j \Delta_j \right) \left\langle \prod_{i=1}^N \Phi_i(z_i) \right\rangle = 0. \quad (2.16)$$

In analogous way, the Ward identity associated to the conserved current  $\mathcal{W}(z)$  takes the form

$$\left\langle \mathcal{W}(z) \prod_{i=1}^N \Phi_i(z_i) \right\rangle = \sum_{i=1}^N \left( \frac{q_i}{(z - z_i)^3} + \frac{W_{-1}^{(i)}}{(z - z_i)^2} + \frac{W_{-2}^{(i)}}{z - z_i} \right) \left\langle \prod_{i=1}^N \Phi_i(z_i) \right\rangle, \quad (2.17)$$

where  $W_{-n}^{(i)}$  is the mode  $W_{-n}$  applied to the field  $\Phi_i$ . From the asymptotic condition  $\lim_{z \rightarrow \infty} \mathcal{W}(z) \rightarrow z^{-6}$  one obtains additional five Ward identities in  $\mathcal{W}_3$  case,

$$\sum_{j=0}^N W_{-2}^{(j)} \left\langle \prod_{i=1}^N \Phi_i(z_i) \right\rangle = 0, \quad (2.18)$$

$$\sum_{j=0}^N \left( z_j W_{-2}^{(j)} + W_{-1}^{(j)} \right) \left\langle \prod_{i=1}^N \Phi_i(z_i) \right\rangle = 0, \quad (2.19)$$

$$\sum_{j=0}^N \left( z_j^2 W_{-2}^{(j)} + 2z_j W_{-1}^{(j)} + q_j \right) \left\langle \prod_{i=1}^N \Phi_i(z_i) \right\rangle = 0, \quad (2.20)$$

$$\sum_{j=0}^N \left( z_j^3 W_{-2}^{(j)} + 3z_j^2 W_{-1}^{(j)} + 3z_j q_j \right) \left\langle \prod_{i=1}^N \Phi_i(z_i) \right\rangle = 0, \quad (2.21)$$

$$\sum_{j=0}^N \left( z_j^4 W_{-2}^{(j)} + 4z_j^3 W_{-1}^{(j)} + 6z_j^2 q_j \right) \left\langle \prod_{i=1}^N \Phi_i(z_i) \right\rangle = 0. \quad (2.22)$$

## 2.2 Highest-weight modules

The representation theory of the  $\mathcal{W}_3$  algebra is strictly related to the classical Lie algebra  $\mathfrak{sl}_3$ . We start this section with some facts about  $\mathfrak{sl}_3$  algebra that are relevant for further discussion.

**$\mathfrak{sl}_3$  modules.** The Lie algebra  $\mathfrak{sl}_3$  is defined by the Cartan matrix  $A$ ,

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad (2.23)$$

the elements of which are the scalar products,  $A_{ij} = \langle \vec{e}_i, \vec{e}_j \rangle$ , of the two simple roots  $\vec{e}_1, \vec{e}_2$ .  $A_{11} = A_{22} = 2$  and  $A_{12} = A_{21} = -1$ . The Weyl vector  $\vec{\rho}$  is half the sum of the positive roots  $\{\vec{e}_1, \vec{e}_2, \vec{e}_1 + \vec{e}_2\}$ , that is,

$$\vec{\rho} = \left( \vec{e}_1 + \vec{e}_2 \right) \quad (2.24)$$

The fundamental weights  $\vec{\omega}_1, \vec{\omega}_2$  form a basis dual to the simple roots one

$$\langle \vec{\omega}_i, \vec{e}_j \rangle = \delta_{ij} \quad (2.25)$$

The irreducible finite-dimensional representations of  $\mathfrak{sl}_3$  are parametrized by a highest-

weight on the lattice  $\mathbb{N}\vec{\omega}_1 + \mathbb{N}\vec{\omega}_2$  spanned by the fundamental weights. In particular,  $\vec{\omega}_1$  is the highest-weight of the fundamental representation of  $\mathfrak{sl}_3$ , with weights  $\vec{h}_i$ ,  $i = 1, 2, 3$ ,

$$\vec{h}_1 = \vec{\omega}_1, \quad \vec{h}_2 = -\vec{\omega}_1 + \vec{\omega}_2, \quad \vec{h}_3 = -\vec{\omega}_2, \quad (2.26)$$

A generic  $\mathfrak{sl}_3$  representation, of highest-weight  $\vec{\Lambda}$ , is indexed by two non-negative integers  $(\lambda_1, \lambda_2)$  such that

$$\vec{\Lambda} = \lambda_1 \vec{\omega}_1 + \lambda_2 \vec{\omega}_2, \quad (2.27)$$

or equivalently by Young diagram  $(1^{\lambda_1}, 2^{\lambda_2})$ , with  $\lambda_k$  columns of length  $k$ , ( $k = 1, 2$ ). The weights of  $\vec{\Lambda}$  are obtained by filling the cells of the Young diagram with integers 1, 2, 3, in non-decreasing order along the rows, and in strictly-increasing order along the columns, in all possible manners. Each weight is then obtained by associating a cell filled with the integer  $i$  to  $\vec{h}_i$  and summing over all the cells of the Young diagram. The number of the diagrams associated with a given weight defines the multiplicity of this weight. For instance, the weights of the fundamental representation are

$$\begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{3} \\ \vec{\omega}_1 & -\vec{\omega}_1 + \vec{\omega}_2 & -\vec{\omega}_2 \end{array} \quad (2.28)$$

There is no multiplicity in this case. In the adjoint representation, the highest-weight vector is  $\vec{\Lambda} = \vec{\omega}_1 + \vec{\omega}_2$ , and the weights are labelled by Young tableaux as,

$$\begin{array}{cccc} \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \\ \vec{\omega}_1 + \vec{\omega}_2 & -\vec{\omega}_1 + 2\vec{\omega}_2 & \vec{0} & \vec{0} \\ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array} \\ 2\vec{\omega}_1 - \vec{\omega}_2 & \vec{\omega}_1 - 2\vec{\omega}_2 & -2\vec{\omega}_1 + \vec{\omega}_2 & -\vec{\omega}_1 - \vec{\omega}_2 \end{array} \quad (2.29)$$

We note that the weight  $\vec{0}$  has multiplicity two.

**$\mathcal{W}_3$  primary operators.** Each primary field  $\Phi_{h,q}(z)$  can be labelled by a vector  $\vec{\alpha}$  in the space spanned by the fundamental  $\mathfrak{sl}_3$  weights,

$$\vec{\alpha} = \alpha_1 \vec{\omega}_1 + \alpha_2 \vec{\omega}_2 \quad (2.30)$$

The vector charge  $\vec{\alpha}$  can be written in terms of the vector charge  $\vec{P}$  as

$$\vec{\alpha} = \vec{P} + Q\vec{\rho} \quad (2.31)$$



where  $\vec{\rho}$  has been defined in (2.24). Introducing the parameters

$$x_i = \vec{P} \cdot \vec{h}_i, \quad i = 1, 2, 3, \quad (2.32)$$

where  $\vec{h}_i$  is defined in (2.26), the quantum numbers,  $h$  and  $q$ , are

$$h = Q^2 + x_1x_2 + x_1x_3 + x_2x_3 = Q^2 - \frac{1}{2} \left( x_1^2 + x_2^2 + x_3^2 \right), \quad (2.33)$$

$$q = i x_1x_2x_3, \quad (2.34)$$

The above expressions are invariant under the  $\mathfrak{sl}_3$  Weyl group action. This group is composed by six elements  $\hat{s}(\vec{P})$  which act on  $\vec{P} = (P_1, P_2)$  in the following way,

$$\begin{aligned} (P_1, P_2) &\rightarrow (-P_1, P_1 + P_2) \rightarrow (P_2, -P_1 - P_2) \rightarrow \\ &\rightarrow (-P_2, -P_1) \rightarrow (P_1 + P_2, -P_2) \rightarrow (-P_1 - P_2, P_1) \end{aligned} \quad (2.35)$$

Henceforth, we will use equivalently the pair  $(h, q)$  or the symbol  $\vec{\alpha}$  to indicate a primary field. As the charge  $\vec{\alpha}$  and its Weyl reflections  $\vec{\alpha} - Q\vec{\rho}$  correspond to the same pair  $(h, q)$ , the notation

$$\Phi_{h,q}(z) \equiv \Phi_{\vec{\alpha}}(z) \equiv \Phi_{Q\vec{\rho} + \hat{s}(\vec{\alpha} - Q\vec{\rho})}(z) \quad (2.36)$$

indicate the same primary field. We will use also the notation:

$$\Phi_{\vec{\alpha}}^*(z) \equiv \Phi_{h,-q}(z) \equiv \Phi_{2Q\vec{\rho} - \vec{\alpha}}(z) \quad (2.37)$$

to indicate the dual field  $\Phi_{\vec{\alpha}}^*$  which satisfies

$$\langle \Phi_{\vec{\alpha}} | \Phi_{\vec{\alpha}} \rangle = \lim_{z \rightarrow \infty} z^{2h} \langle \Phi_{\vec{\alpha}}^*(z) \Phi_{\vec{\alpha}}(0) \rangle = 1. \quad (2.38)$$

The above relation fixes the normalization of the primary operators of the theory. Note that the vector  $2Q\vec{\rho} - \vec{\alpha}$  is associated to  $\vec{P} = (-P_1, -P_2)$ .

The charge parametrization of the primary operator is reminiscent of their Coulomb gas representation, introduced in Section 2.47. In particular, when one represents a primary operator with a vertex operator in a Coulomb gas theory, the identification (2.36) implies non-trivial normalisations factors, *cf* (2.44).

**Structure constants of the diagonal  $\mathcal{W}_3$  theory.** The symmetry group of the  $\mathcal{W}_3$  theory is the tensor product  $\mathcal{W}_3 \otimes \bar{\mathcal{W}}_3$  of holomorphic and anti-holomorphic algebras. The spectrum of the diagonal theory, which is built from the  $\mathcal{W}_3 \otimes \bar{\mathcal{W}}_3$  representations, is composed by the primary fields

$$\Phi_{\vec{\alpha}}^{\text{phys}}(z, \bar{z}) \equiv \Phi_{\vec{\alpha}}(z)\Phi_{\vec{\alpha}}(\bar{z}). \quad (2.39)$$

The operator product expansion, OPE, of two primary fields takes the form

$$\Phi_{\vec{\alpha}_M}^{\text{phys}}(z, \bar{z})\Phi_{\vec{\alpha}_R}^{\text{phys}}(0) = \sum_L |z|^{-2h_R-2h_M+2h_L} C_{\vec{\alpha}_M, \vec{\alpha}_R}^{\vec{\alpha}_L} \left[ \Phi_{\vec{\alpha}_L}^{\text{phys}}(z, \bar{z}) \right], \quad (2.40)$$

where  $\left[ \Phi_{\vec{\alpha}_L}^{\text{phys}}(z, \bar{z}) \right]$  denotes the contribution of the primary field and all its descendants and the summation in the above equation goes over all the possible primaries  $\Phi_L^{\text{phys}}$  (that can constitute a continuous set). The structure constants  $C_{\vec{\alpha}_M, \vec{\alpha}_R}^{\vec{\alpha}_L}$  can be defined by the 3-point function,

$$C_{\vec{\alpha}_M, \vec{\alpha}_R}^{\vec{\alpha}_L} = \left\langle \Phi_L^{\text{phys}} \left| \Phi_M^{\text{phys}}(1) \Phi_R^{\text{phys}}(0) \right. \right\rangle. \quad (2.41)$$

**Fully-degenerate representations.** Each  $\mathcal{W}_3$  fully-degenerate representation is associated with a primary field  $\Phi_{\vec{\alpha}}(z)$ , with a vector-valued charge  $\vec{\alpha}$ ,

$$\vec{\alpha}_{r_1 r_2 s_1 s_2} = b \left( (1 - r_1) \vec{\omega}_1 + (1 - r_2) \vec{\omega}_2 \right) + \frac{1}{b} \left( (1 - s_1) \vec{\omega}_1 + (1 - s_2) \vec{\omega}_2 \right), \quad (2.42)$$

where  $r_1, r_2, s_1, s_2$  are positive integers. We denote this primary field by  $\Phi_{r_1 r_2 s_1 s_2}(z)$ . The highest-weight representation  $\mathcal{V}_{r_1 r_2 s_1 s_2}$  associated to the field  $\Phi_{r_1 r_2 s_1 s_2}$  exhibits two independent null-states with charges  $\vec{\alpha}_{r_1, r_2, s_1, s_2} + b r_1 \vec{e}_1$  and  $\vec{\alpha}_{r_1, r_2, s_1, s_2} + b r_2 \vec{e}_2$ . The fusion products of  $\mathcal{V}_{r_1 r_2 s_1 s_2}$  with a general  $\mathcal{W}_3$  irreducible module  $\mathcal{V}_{\vec{\alpha}}$  takes the form

$$\mathcal{V}_{r_1 r_2 s_1 s_2} \times \mathcal{V}_{\vec{\alpha}} = \sum_{\vec{h}_r, \vec{h}_s} \mathcal{V}_{\alpha - b \vec{h}_r - b^{-1} \vec{h}_s}, \quad (2.43)$$

where  $h_r$  and  $h_s$  are the weights of the  $\mathfrak{sl}_3$  representation with highest-weight  $(r_1 - 1) \vec{\omega}_1 + (r_2 - 2) \vec{\omega}_2$  and  $(s_1 - 1) \vec{\omega}_1 + (s_2 - 2) \vec{\omega}_2$  respectively.

### 2.3 The Coulomb gas approach

We now briefly describe the Coulomb gas approach to  $\mathcal{W}_3$  Toda theory, which will be relevant in constructing conformal blocks in Section 5.

In the Coulomb gas approach, a primary field  $\Phi_{\vec{\alpha}}^{\text{phys}}$  is represented by the exponential field  $V_{\vec{\alpha}}^{\text{phys}}(z, \bar{z}) =: e^{\vec{\alpha} \cdot \vec{\phi}(z, \bar{z})} :$ , where  $\vec{\phi}$  is a two-component free boson field. In particular, we have the identification

$$\Phi_{\vec{\alpha}}^{\text{phys}} = N_{\vec{\alpha}}^{-1} V_{\vec{\alpha}}(z, \bar{z}), \quad (2.44)$$

where the normalisation constant  $N_{\vec{\alpha}}^{-1}$  has been computed in [13]. For a given  $\vec{\alpha}$ , several vertex operators can be used, since we have the identification

$$V_{\vec{\alpha}} = R_{s(\vec{\alpha})} V_{s(\vec{\alpha})}, \quad (2.45)$$

where  $R_{s(\vec{\alpha})}$  is the *reflection amplitude* in Toda theory [13]. The free boson correlation function is,

$$\left\langle \prod_{i=1}^N V_{\vec{\alpha}_i}^{\text{phys}}(z_i, \bar{z}_i) \right\rangle_{\text{free boson}} = \delta\left(\sum_{i=1}^N \vec{\alpha}_i, 2Q\vec{\rho}\right) \prod_{1 \leq i < j \leq N} |z_i - z_j|^{-4\vec{\alpha}_i \cdot \vec{\alpha}_j}, \quad (2.46)$$

where  $\delta_{a,b}$  is the Kronecker delta and the background charge  $2Q\vec{\rho}$  is at infinity. This method provides the following integral representation for the  $\mathcal{W}_3$  correlation functions on the plane,

$$\left\langle \prod_{i=1}^N \Phi_{\vec{\alpha}_i}^{\text{phys}}(z_i, \bar{z}_i) \right\rangle \propto \left\langle \mathcal{Q}_1^{n_1}(b) \mathcal{Q}_1^{m_1}(b^{-1}) \mathcal{Q}_2^{n_2}(b) \mathcal{Q}_2^{m_2}(b^{-1}) \prod_{i=1}^N V_{\vec{\alpha}_i}^{\text{phys}}(z_i, \bar{z}_i) \right\rangle_{\text{free boson}}, \quad (2.47)$$

where  $\mathcal{Q}_{1,2}(b)$  are screening operators

$$\mathcal{Q}_k(b) = \int e^{b\vec{e}_k \cdot \vec{\phi}(z, \bar{z})} d^2 z, \quad (2.48)$$

if the neutrality condition

$$\sum_{i=1}^N \vec{\alpha}_i + \left(n_1 b + m_1 b^{-1}\right) \vec{e}_1 + \left(n_2 b + m_2 b^{-1}\right) \vec{e}_2 = 2Q\vec{\rho}, \quad (2.49)$$

is fulfilled for some natural  $n_k, m_k$ ,  $k = 1, 2$ .

A Coulomb gas approach can be used to determine directly the holomorphic components of the correlation function, that is, the conformal block. We refer the reader to Chapter 8 of [14] for a detailed description of this approach. In this case, one formally considers the holomorphic part of the exponential field  $V_{\vec{\alpha}}^{\text{phys}}(z, \bar{z}) \rightarrow V_{\vec{\alpha}}(z)$ , and associates the holomorphic primary field to it,  $\Phi_{\vec{\alpha}_i} = V_{\vec{\alpha}_i}(z)$ . The exponential free boson correlator reads,

$$\left\langle \prod_{i=1}^N V_{\vec{\alpha}_i}(z_i) \right\rangle_{\text{free boson}} = \delta\left(\sum_{i=1}^N \vec{\alpha}_i, 2Q\vec{\rho}\right) \prod_{1 \leq i < j \leq N} (z_i - z_j)^{-2\vec{\alpha}_i \cdot \vec{\alpha}_j}. \quad (2.50)$$

In the case the neutrality condition (2.49) is satisfied, one obtains the representation for the conformal block in terms of closed contour integrals,

$$\left\langle \prod_{i=1}^N \Phi_{\vec{\alpha}_i}(z_i) \right\rangle \propto \left\langle \prod_{i=1,2} \left( \oint du V_{b\vec{e}_i}(u) \right)^{n_i} \left( \oint du V_{b^{-1}\vec{e}_i}(u) \right)^{m_i} \prod_{i=1}^N V_{\vec{\alpha}_i}(z_i) \right\rangle_{\text{free boson}} \quad (2.51)$$

The different choices of the contours are in correspondence with the different fusion channels, as shown in our specific case below. For more details on Coulomb gas approach in  $\mathfrak{sl}_3$  Toda theory, we refer the reader to [1, 5, 15].

### 3 Conformal blocks and matrix elements

In the following, we define a general  $\mathcal{W}_3$  4-point conformal block.

#### 3.1 Power series expansion of the conformal block

Using invariance under global conformal transformations (2.14–2.16), we fix  $z_R = 0$ ,  $z_2 = 1$ , and  $z_L = \infty$ , and consider the 4-point conformal block  $\mathcal{B}_M \left( L, 2, 1, R \right) (z)$  as a function of the holomorphic coordinate  $z_1 = z$ . The fusion channel is labelled  $M$  in the *comb diagram*,

$$\mathcal{B}_M \left( L, 2, 1, R \right) (z) \equiv \langle \Phi_L | \Phi_2(1) \Phi_1(z) | \Phi_R \rangle \equiv \Phi_L^*(\infty) \begin{array}{c} \Phi_2(1) \quad \Phi_1(z) \\ | \quad | \\ \hline M \\ \hline \Phi_R(0) \end{array} \quad (3.1)$$

The most general  $\mathcal{W}_3$  4-point conformal block depends on ten parameters, since each primary field, as well as the highest-weight representation that flows in the fusion channel, is specified by two quantum numbers,  $h$  and  $q$ . In the presence of multiplicities, the ten parameters are not sufficient to determine the conformal block, as we explain later on our specific example (see the discussion below (5.33)). Using standard techniques, one can express the conformal block as a series expansion in  $z$ ,

$$z^{h_1+h_R-h_M} \mathcal{B}_M \left( L, 2, 1, R \right) (z) = 1 + \sum_{i=1}^{\infty} z^i \sum_{\substack{K_M, K'_M \\ |K_M|=|K'_M|=i}} \left( H^{-1} \right)_{K_M, K'_M} \Gamma_{\emptyset_L, \emptyset_2, K_M} \Gamma'_{K'_M, \emptyset_1, \emptyset_R}, \quad (3.2)$$

where the main ingredients are the elements of the Shapovalov matrix of inner products,  $H$ , and the matrix elements of the primary fields between two arbitrary descendant states,  $\Gamma_{I_L, J_M, K_R}$  and  $\Gamma'_{I_L, J_M, K_R}$ .

### 3.2 The Shapovalov matrix of inner products

The Shapovalov matrix  $H$ , whose  $ij$ -element  $H_{ij}$  is the scalar product of the states  $|I\rangle_{h,q} = \mathcal{L}_I|h, q\rangle$  and  $|J\rangle_{h,q} = \mathcal{L}_J|h, q\rangle$ , defined in terms of the basis (2.11),

$$H_{IJ} = {}_{h,q}\langle I|J\rangle_{h,q}. \quad (3.3)$$

$H$  is a block-diagonal matrix,  $\text{diag} \left( H_0, H_1, H_2, \dots \right)$ , where the elements of the  $i$ -th block,  $H_i$ , are the scalar products of the level- $i$  descendants. These elements can be computed using the commutation relations (2.6). By definition,  $H_0 = 1$ , and the explicit forms of  $H_1$  and  $H_2$  are given in Appendix A.

### 3.3 The matrix elements

The second ingredient in (3.2) is the matrix elements. Using the notation of [6], we are interested in the matrix elements of general descendant fields

$$\Gamma_{I_L, J_M, K_R} = \frac{\langle (\Phi_L^*)^{(I)_L} | \Phi_M^{(J)_M}(1) \Phi_R^{(K)_R}(0) \rangle}{\langle (\Phi_L^*) | \Phi_M(1) \Phi_R(0) \rangle}, \quad (3.4)$$

$$\Gamma'_{I_L, J_M, K_R} = \frac{\langle \Phi_L^{(I)_L} | \Phi_M^{(J)_M}(1) \Phi_R^{(K)_R}(0) \rangle}{\langle \Phi_L | \Phi_M(1) \Phi_R(0) \rangle}, \quad (3.5)$$

where the descendants fields  $\Phi_X^{(I)^X} = \mathcal{L}_I \Phi_X$ ,  $X = L, M, R$ , were defined in (2.11). The matrix elements  $\Gamma$  and  $\Gamma'$  are related by changing the sign of  $q_L$ .

To compute the 4-point conformal block explicitly, we need to express each of these matrix elements in terms of matrix elements of three primary fields. This can be achieved using Virasoro and  $\mathcal{W}_3$  Ward identities in a systematic way, as explained in [4, 6, 16–19] or in the work of Kanno *et al.* [6], where an algorithm to compute the matrix elements of descendant fields in terms of the matrix elements of the corresponding primaries, was outlined. We review below this procedure

Let us start with the Virasoro Ward identities. These lead to three recursion relations for the matrix elements. For  $n \in \mathbb{Z} > 0$ , one of these recursion relations is

$$\begin{aligned}
\langle L_{-n} \Phi_L^{(I)L} | \Phi_M^{(J)M} | \Phi_R^{(K)R} \rangle = \\
\langle L_0 \Phi_L^{(I)L} | \Phi_M^{(J)M} | \Phi_R^{(K)R} \rangle + \langle \Phi_L^{(I)L} | \Phi_M^{(J)M} | \left( L_n - L_0 \right) \Phi_R^{(K)R} \rangle + \\
\langle \Phi_L^{(I)L} | \left( nL_0 + \sum_{i=1}^n \frac{(n+1)!}{(i+1)!(n-i)!} L_i \right) \Phi_M^{(J)M} | \Phi_R^{(K)R} \rangle, \quad (3.6)
\end{aligned}$$

The point of the recursion relation (3.6) is that the Virasoro creation operator  $L_{-n}$ ,  $n > 0$ , that acts on  $\Phi_L^{(I)L}$  on the left hand side of (3.6), is replaced by  $L_0$ , and annihilation operators  $L_i$ ,  $i > 0$ , that act on  $\Phi_M^{(J)M}$  and  $\Phi_R^{(K)R}$  on the right hand side. Since the level of the initial descendant  $L_{-n} \Phi_L^{(I)L}$ , on the left hand side is finite, the final descendants on the right hand side can be expanded in terms of basis states, in finitely-many steps, as explained in E.

The result of the application of (3.6) is that the initial matrix element on the left hand side is expanded in terms of new matrix elements on the right hand side. The new matrix elements are such that each of them contains one descendant that is *closer* to its primary state, while the other two are the same. Thus each of the matrix elements on the right hand side are closer to matrix elements of primaries. Applying (3.6) repeatedly, we can compute the matrix element of any descendant on the left hand side in terms of the matrix element of the corresponding primaries. Most importantly, we can do that in finitely many steps. In each step, we peel a Virasoro creation operator off a descendant, and replace it with  $L_0$  or an annihilation operator, and thereby bring the matrix element closer to that of three primaries. The two remaining Virasoro recursion relations of Kanno *et al.* are analogous to (3.6), but involve peeling a Virasoro creation operator  $L_{-n}$ ,  $n > 0$ , off  $\Phi_M^{(J)M}$  and off  $\Phi_R^{(K)R}$ , respectively. They can be found in [6].

Next, we turn to the  $\mathcal{W}_3$  Ward identities. These also lead to three recursion relations for the matrix elements. One of these is

$$\begin{aligned}
\langle W_{-n} \Phi_L^{(I)L} | \Phi_M^{(J)M} | \Phi_R^{(K)R} \rangle = \\
\langle W_0 \Phi_L^{(I)L} | \Phi_M^{(J)M} | \Phi_R^{(K)R} \rangle + \langle \Phi_L^{(I)L} | \Phi_M^{(J)M} | \left( W_n - W_0 \right) \Phi_R^{(K)R} \rangle + \\
\langle \Phi_L^{(I)L} | \left( nW_{-1} + \frac{1}{2}n(n+3)W_0 + \sum_{i=1}^n \frac{(n+2)!}{(i+2)!(n-i)!} W_i \right) \Phi_M^{(J)M} | \Phi_R^{(K)R} \rangle, \quad (3.7)
\end{aligned}$$

and two more  $\mathcal{W}_3$  recursion relations that are analogous to (3.7), but involve peeling a  $\mathcal{W}_3$  creation operator  $W_{-n}$ ,  $n > 0$ , off  $\Phi_M^{(J)M}$  and  $\Phi_R^{(K)R}$ , respectively [6]. The result of the algorithm of Kanno *et al.* in the  $\mathcal{W}_3$  case, is not as simple as in the Virasoro case. In the  $\mathcal{W}_3$  case, one can start from any matrix element of three arbitrary  $\mathcal{W}_3$  descendant states,  $\Phi_L^{(I)L}$ ,  $\Phi_M^{(J)M}$ , and  $\Phi_R^{(K)R}$ , and re-write it in terms of the matrix element

$$\langle \Phi_L | \left( W_{-1}^p \Phi_M \right) | \Phi_R \rangle, \quad p = 0, 1, 2, \dots \quad (3.8)$$

the evaluation of which, for arbitrary  $\Phi_M$ , is non-trivial, and generally not explicitly known.

**Matrix elements of the semi-degenerate fundamental field.** A primary field  $\Phi_{\vec{\alpha}}(z)$  in the fundamental representation of  $\mathfrak{sl}_3$  has a charge  $\vec{\alpha}$  that is collinear to  $\vec{\omega}_1$ ,

$$\vec{\alpha} = a \vec{\omega}_1, \quad (3.9)$$

and quantum numbers

$$h_a = \frac{a}{3b} \left( 3 + 3b^2 - ab \right), \quad q_a = \frac{ia}{27b^2} \left( 3 + 3b^2 - ab \right) \left( 3 + 3b^2 - 2ab \right). \quad (3.10)$$

We consider first the case where the fusion field  $\Phi_M$  in the above matrix elements is associated to the fundamental representation,

$$\Phi_M = \Phi_{a\vec{\omega}_1}. \quad (3.11)$$

It can be shown that the corresponding highest-weight representation has a null-state at level-one, so that in general  $\Phi_{a\vec{\omega}_1}$  is semi-degenerate. Imposing the vanishing of the null-state, one obtains the identity

$$W_{-1} \Phi_{a\vec{\omega}_1} = \frac{3q_a}{2h_a} L_{-1} \Phi_{a\vec{\omega}_1}. \quad (3.12)$$

In [6], Kanno *et al.* use this identity to compute the matrix elements (3.8) of  $\Phi_M$ , defined as in (3.11). The procedure is as follows. We replace the string  $W_{-1}^p$  by the string  $W_{-1}^{p-1} L_{-1}$ , expand the operator  $W_{-1}^{p-1} L_{-1}$  in terms of the basis (2.11), then apply the algorithm to peel all Virasoro creation operators,  $L_{-n}, n > 0$ , and all  $\mathcal{W}_3$  creation operators,  $W_{-n}, n > 1$ , that act on  $\Phi_M$  in the resulting matrix elements, and re-express these as diagonal and annihilation operators on  $\Phi_L$  or  $\Phi_R$ . The action of the diagonal operators on the primary fields is known, and the action of the annihilation operators is zero by definition. Repeating this procedure, Kanno *et al.* express any matrix element of a primary field with a highest-weight in the fundamental of  $\mathfrak{sl}_3$ , in terms of the matrix element of the corresponding primaries. For instance,

$$\langle \Phi_L | \left( W_{-1} \Phi_{a\vec{\omega}_1} \right) | \Phi_R \rangle = \frac{3}{2} \frac{q_a}{h_a} \left( -h_a + h_L - h_R \right) \langle \Phi_L | \Phi_{a\vec{\omega}_1} | \Phi_R \rangle, \quad (3.13)$$

and

$$\begin{aligned} \langle \Phi_L | \left( W_{-1}^2 \Phi_{a\bar{\omega}_1} \right) | \Phi_R \rangle = \\ \frac{3q_a}{2h_a} \left( \frac{3q_a}{2h_a} (-h_a + h_L - h_R)(-h_a + h_L - h_R - 3) - q_L - q_R - q_a \right) \\ \langle \Phi_L | \Phi_{a\bar{\omega}_1} | \Phi_R \rangle. \end{aligned} \quad (3.14)$$

For the purposes of the present work, we will use this algorithm in the case  $a = -b$ . This corresponds to identifying the fusion field  $\Phi_M$  with the fully-degenerate fundamental field  $\Phi_{-b\bar{\omega}_1} = \Phi_{2111}$ . In addition to the level-1 null-state (3.12), this field also obeys level-2 and level-3 null-state conditions,

$$W_{-2}\Phi_{-b\bar{\omega}_1} = \frac{1}{\sqrt{\eta}} \left( \frac{12q}{h(5h+1)} L_{-1}^2 - \frac{6q(h+1)}{h(5h+1)} L_{-2} \right) \Phi_{-b\bar{\omega}_1}, \quad (3.15)$$

and

$$\begin{aligned} W_{-3}\Phi_{-b\bar{\omega}_1} = \frac{1}{\sqrt{\eta}} \left( \frac{16q}{h_M(h+1)(5h+1)} L_{-1}^3 \right. \\ \left. - \frac{12q}{h(5h+1)} L_{-1} L_{-2} - \frac{3q(h-3)}{2h(5h+1)} L_{-3} \right) \Phi_{-b\bar{\omega}_1}, \end{aligned} \quad (3.16)$$

where  $h$  and  $q$  are given respectively by  $h_a$  and  $q_a$  in the (3.10) with  $a = -b$ ,

$$h = h_{2111} = \frac{1}{3} \left( -3 - 4b^2 \right), \quad q = q_{2111} = -\frac{i}{27b} \left( 3 + 4b^2 \right) \left( 3 + 5b^2 \right) \quad (3.17)$$

and  $\eta$  has been defined in (2.8).

**Matrix elements of the fully-degenerate adjoint field.** We are interested in computing the matrix elements, when the primary field  $\Phi_M$ , that flows in the fusion channel, is the fully-degenerate primary field  $\Phi_{2211}$ <sup>1</sup>,

$$\Phi_M = \Phi_{2211}, \quad (3.18)$$

with quantum numbers,

$$h = h_{2211} = -2 - 3b^2, \quad q = q_{2211} = 0. \quad (3.19)$$

The  $\Phi_{2211}$  highest-weight representation has two null-states at level-two. Considering the states at level-two, we obtain, see Appendix C, the null-state conditions,

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<sup>1</sup> The result for the primary  $\Phi_{1122}$  is obtained from the result for  $\Phi_{2211}$  by replacing  $b \rightarrow \frac{1}{b}$ .



$$L_{-1}W_{-1}\Phi_{2211} = \frac{1}{2}(h+1)W_{-2}\Phi_{2211}, \quad (3.20)$$

and

$$W_{-1}^2\Phi_{2211} = \frac{1}{\eta} \left( -\frac{2(h+2)(h+1)h}{(h-3)(5h+1)}L_{-2} + \frac{3(h+1)(h-1)}{2(h-3)(5h+1)}L_{-1}^2 \right) \Phi_{2211}. \quad (3.21)$$

From (3.21), the action of  $W_{-1}^2$  on  $\Phi_{2211}$  can be re-written in terms of the action of Virasoro operators. In analogy with the fundamental case, using the algorithm of Kanno *et al.*, together with the identity (3.21), one can compute all the matrix elements of  $\Phi_{2211}$ . More specifically, using the algorithm, one can re-write any matrix element  $\langle \Phi_L^{(I)} | \Phi_{2211} | \Phi_R^{(K)} \rangle$  in terms of

$$\langle \Phi_L | \left( W_{-1}^p \Phi_{2211} \right) | \Phi_R \rangle, \quad p = 0, 1, 2, \dots, \quad (3.22)$$

Next, one applies (3.21) to reduce the degree  $p$  to the degree  $(p-2)$ . In the case  $p=2$ , one obtains

$$\begin{aligned} \langle \Phi_L | \left( W_{-1}^2 \Phi_{2211}(1) \right) | \Phi_R \rangle = \\ \frac{1}{\eta} \left( \frac{3(h+1)(h-1)(-h+h_L-h_R)(-h+h_L-h_R-1)}{2(h-3)(5h+1)} \right. \\ \left. - \frac{2h(h+1)(h+2)(h-h_L+2h_R)}{(h-3)(5h+1)} \right) \langle \Phi_L | \Phi_{2211} | \Phi_R \rangle. \end{aligned} \quad (3.23)$$

When  $p > 2$ , one needs to apply (3.21) finitely-many times. Following every application, one obtains two matrix elements of states that are not in basis form. These states need to be expanded in terms of basis states (2.11). Following that, one applies the algorithm of Kanno *et al.* to each of these states. Since  $p$  is initially finite, one obtains, in finitely-many steps, a matrix element of three primaries, or a linear combination of the latter and the matrix element (3.22), but now with  $p=1$ . Using the relation (3.20) together with the identities (2.14)-(2.22), the latter can be explicitly evaluated as,

$$\langle \Phi_L | \left( W_{-1} \Phi_{2211} \right) | \Phi_R \rangle = \frac{(q_R - q_L)(h+1)}{2(h_R - h_L)} \langle \Phi_L | \Phi_{2211} | \Phi_R \rangle. \quad (3.24)$$

When  $\Phi_L = \Phi_R$  the matrix element

$$\left\langle \Phi_L | \left( W_{-1} \Phi_{2211} \right) | \Phi_L \right\rangle \quad (3.25)$$

is not defined. This matrix element is related to the channel  $\vec{\alpha}_L = \vec{\alpha}_R + \vec{0}$  and therefore to the two-fold degenerate  $\vec{0}$  weight in the adjoint representation. The same phenomenon

happens in the classical  $\mathfrak{sl}_3$  representation theory where the  $SU(3)$  Clebsch-Gordan coefficients are not uniquely defined when multiplicities are present (see for instance [20]).

## 4 A fourth-order Fuchsian differential equation

We consider the 4-point conformal block

$$\mathcal{B}_M(z) \equiv \Phi_L^*(\infty) \text{---} \overset{\Phi_{2211}(1) \quad \Phi_{2111}(z)}{\underset{M}{\text{---}}} \Phi_R(0), \quad (4.1)$$

where, for the moment,  $\Phi_L$  and  $\Phi_R$  are general fields. Using the null-state conditions associated with  $\Phi_{2211}(1)$ , together with the null-state conditions associated with  $\Phi_{2111}(z)$ , we derive a fourth-order Fuchsian differential equation for  $\mathcal{B}_M(z)$ . Since this is a standard procedure (see, for example, [5, 21, 22]), it suffices to outline the derivation. For sake of clarity, it is convenient to re-introduce the explicit dependence of the conformal block on general positions  $z_i$ ,  $i = 1, 2, 3, 4$  of the external operators,

$$\mathcal{B}_M(\{z_i\}) = z_{41}^{-2h_1} z_{42}^{h_1+h_3-h_2-h_4} z_{43}^{h_1+h_2-h_3-h_4} z_{32}^{h_4-h_1-h_2-h_3} \mathcal{B}_M \left( \frac{z_{12}z_{34}}{z_{14}z_{32}} \right), \quad (4.2)$$

where  $z_{ij} = z_i - z_j$ . The operators  $\Phi_{2111}$ ,  $\Phi_{2211}$ ,  $\Phi_R$  and  $\Phi_L$  are placed at positions  $z_1$ ,  $z_2$ ,  $z_3$  and  $z_4$  respectively. The conformal block (4.1) is obtained in the limit  $z_1 \rightarrow z$ ,  $z_2 \rightarrow 0$ ,  $z_3 \rightarrow 1$  and  $z_4 \rightarrow \infty$ . According to these conventions, in the next formula, we use the labelling,

$$h_1 = h_{2111}, \quad q_1 = q_{2111}, \quad h_3 = h_{2211}, \quad q_3 = q_{2211} = 0 \quad (4.3)$$

$$h_2 = h_R, \quad q_2 = q_R, \quad h_4 = h_L, \quad q_4 = -q_L, \quad (4.4)$$

see (3.17) and (3.19).

**The adjoint and fundamental null-state conditions.** First, we analyze the consequences of the null-state condition (3.16). The RHS in (3.16) contains only Virasoro generators and does not represent any difficulty. Using (2.13) it can be easily transformed into the differential operator acting on  $\mathcal{B}_M \left( \{z_i\} \right)$ ,

$$\begin{aligned} \text{RHS} \Rightarrow \frac{1}{\sqrt{\eta}} & \left( \frac{16q_1}{h_1(h_1+1)(5h_1+1)} \partial_{z_1}^3 \right. \\ & - \frac{12q_1}{h_1(5h_1+1)} \partial_{z_1} \sum_{k \neq 1} \left( \frac{h_k}{(z_1 - z_k)^2} + \frac{\partial_{z_k}}{(z_1 - z_k)} \right) \\ & \left. + \frac{3q_1(h_1-3)}{2h_1(5h_1+1)} \sum_{k \neq 1} \left( \frac{2h_k}{(z_1 - z_k)^3} + \frac{\partial_{z_k}}{(z_2 - z_k)^2} \right) \right) \mathcal{B}_M \left( \{z_i\} \right). \quad (4.5) \end{aligned}$$

For the LHS in (3.16) we can use (2.17) to get

$$\text{LHS} \Rightarrow \sum_{k \neq 1} \left( \frac{q_k}{(z_1 - z_k)^3} + \frac{W_{-1}^{(k)}}{(z_1 - z_k)^2} + \frac{W_{-2}^{(k)}}{(z_1 - z_k)} \right) \mathcal{B}_M \left( \{z_i\} \right). \quad (4.6)$$

Now we have to take into account the  $\mathcal{W}_3$  identities (2.18–2.22). These five relations mix nine unknown functions,  $W_{-1}^{(i)} \mathcal{B}_M \left( \{z_i\} \right)$ ,  $i = 1, \dots, 4$ ,  $W_{-2}^{(i)} \mathcal{B}_M \left( \{z_i\} \right)$ ,  $i = 1, \dots, 4$  and  $\mathcal{B}_M \left( \{z_i\} \right)$ . The fact that we have at position  $z_1$  the field  $\Phi_{-b\vec{\omega}_1}$  provides three more equations, the level-1 and level-2 conditions (3.12), and (3.15),

$$W_{-1}^{(1)} \mathcal{B}_M \left( \{z_i\} \right) = \frac{3q_1}{2h_1} \partial_{z_1} \mathcal{B}_M \left( \{z_i\} \right), \quad (4.7)$$

$$\begin{aligned} W_{-2}^{(1)} \mathcal{B}_M \left( \{z_i\} \right) = & \left( \frac{12q_1}{h_1(5h_1+1)} \partial_{z_1}^2 - \frac{6q_1(h_1+1)}{h_1(5h_1+1)} \sum_{k \neq 1} \left( \frac{h_k}{(z_1 - z_k)^2} + \frac{\partial_{z_k}}{(z_1 - z_k)} \right) \right) \mathcal{B}_M \left( \{z_i\} \right), \quad (4.8) \end{aligned}$$

and the level-3 equation (3.16) which can be expressed by equating (4.5) and (4.6). Finally, the last equation that we need is provided by the null-state condition (3.20),

$$\partial_{z_3} W_{-1}^{(3)} \mathcal{B}_M \left( \{z_i\} \right) = \frac{1}{2}(h_3+1) W_{-2}^{(3)} \mathcal{B}_M \left( \{z_i\} \right). \quad (4.9)$$

Using the previous relations, we obtained a partial differential equation for  $\mathcal{B}_M(\{z_i\})$ . By fixing projective invariance with help of (2.14–2.16), this can be transformed in a fourth-order Fuchsian equation for  $\mathcal{B}_M(z)$ . The explicit form of the differential equation is given in (4.20) and in (4.21).

**Fusion rules and local exponents.** Before giving the explicit form of the fourth-order Fuchsian equation, we first discuss the general properties of its solutions. Of course, these solutions have to correspond to the ‘*admissible*’ fusion, corresponding to the case when  $\vec{\alpha}_L$  and  $\vec{\alpha}_R$  are related through

$$\vec{\alpha}_L + b\vec{\beta}_1 + b\vec{\beta}_2 = \vec{\alpha}_R, \quad (4.10)$$

where  $\vec{\beta}_1$  is a weight in the fundamental representation, with highest-weight  $\vec{\omega}_1$ , and  $\vec{\beta}_2$  is a weight in the adjoint representation, with highest-weight  $(\vec{\omega}_1 + \vec{\omega}_2)$ . We expect the case  $\vec{\beta}_2 = \vec{0}$ , which corresponds to the multiplicity-2 weight in the adjoint representation to require special attention, so we focus on it. Without loss of generality, we choose  $\vec{\beta}_1 = \vec{\omega}_1$ , the two other cases being related to it by the action of the Weyl group. We end up with,

$$\vec{\alpha}_L + b\vec{\omega}_1 = \vec{\alpha}_R. \quad (4.11)$$

Defining

$$\vec{\alpha}_R = a_{R_1}\vec{\omega}_1 + a_{R_2}\vec{\omega}_2, \quad (4.12)$$

the conformal block  $\mathcal{B}_M(z)$  is a function of three parameters,  $a_{R_1}$ ,  $a_{R_2}$  and  $b$ . The dependence on  $M = 1, 2, 3, 4$ , indexes the four solutions of the fourth-order Fuchsian differential equation. The possible channels are,

$$\text{Channel 1: } \vec{\alpha}_M = \vec{\alpha}_R + b(\vec{\omega}_1 - \vec{\omega}_2) \quad (4.13)$$

$$\text{Channel 2: } \vec{\alpha}_M = \vec{\alpha}_R + b\vec{\omega}_2 \quad (4.14)$$

$$\text{Channel 3 and channel 4: } \vec{\alpha}_M = \vec{\alpha}_R - b\vec{\omega}_1 \quad (4.15)$$

Channels 3 and 4 correspond to the fusion

$$\Phi_{\vec{\alpha}_L} \Phi_{2211} = \Phi_{\vec{\alpha}_R - b\vec{\omega}_1} \Phi_{2211} \rightarrow \Phi_{\vec{\alpha}_R - b\vec{\omega}_1}, \quad (4.16)$$

which reflects the fact that, in the adjoint representation, with highest weight  $(\vec{\omega}_1 + \vec{\omega}_2)$ , there is a weight  $\vec{\Lambda} = \vec{0}$  with multiplicity 2. This is consistent with the local exponents of the Fuchsian differential equation, given below, with singularities at 0, 1 and  $\infty$ . In Riemann-symbol notation<sup>2</sup>, the local exponents are

$$\left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \\ \alpha_3 + 1 & \beta_3 + 1 & \gamma_3 + 1 \end{array} \right\} \quad (4.17)$$

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<sup>2</sup> We refer the reader to [23] for an exhaustive overview of Fuchsian systems.

with

$$\begin{aligned}
\alpha_1 &= -\frac{a_{R_1}}{3}b + \frac{a_{R_2}}{3}b + b^2 + 1, & \alpha_2 &= 2 - \frac{a_{R_1}}{3}b - \frac{2a_{R_2}}{3}b + 2b^2, & \alpha_3 &= \frac{2a_{R_1}}{3}b + \frac{a_{R_2}}{3}b, \\
\beta_1 &= 1 + b^2, & \beta_2 &= 2 + 3b^2, & \beta_3 &= -b^2, \\
\gamma_1 &= -2 + \frac{a_{R_1}}{3}b + \frac{2a_{R_2}}{3}b - 3b^2, & \gamma_2 &= -1 + \frac{a_{R_1}}{3}b - \frac{a_{R_2}}{3}b - 2b^2, & \gamma_3 &= -\frac{2a_{R_1}}{3}b - \frac{a_{R_2}}{3}b.
\end{aligned} \tag{4.18}$$

One can verify that:

$$\sum_{i=1}^4 \left( \alpha_i + \beta_i + \gamma_i \right) = 6, \tag{4.19}$$

where  $\alpha_4 = \alpha_3 + 1$ ,  $\beta_4 = \beta_3 + 1$  and  $\gamma_4 = \gamma_3 + 1$ . This is the Fuchsian relation for a fourth-order Fuchsian differential equation with  $2 + 1$  singularities. We see that the multiplicity in the representation with highest-weight  $\left( \vec{\omega}_1 + \vec{\omega}_2 \right)$  reflects in the degeneracy of characteristic exponents, two of which differ by an integer. This means that we have a 2-dimensional space spanned by two solutions with the same local exponents. In other words, we have a family of solutions (conformal blocks) of the fourth-order differential equation whose expansion at the first-order is not fixed.

**Explicit form of the fourth-order differential equation.** We found that the differential equation obeyed by the conformal block (4.1) can be given in a simple form in terms of the function  $\mathcal{F}(z)$ , defined as,

$$\mathcal{B}_M(z) = z^{\alpha_3}(z-1)^{\beta_3}\mathcal{F}(z), \tag{4.20}$$

For the function  $\mathcal{F}(z)$  we get

$$\begin{aligned}
& z^2(z-1)^2\mathcal{F}''''(z) + z(z-1) \left( b_1z + c_1 \right) \mathcal{F}'''(z) \\
& + \left( a_2z^2 + b_2z + c_2 \right) \mathcal{F}''(z) + \left( b_3z + c_3 \right) \mathcal{F}'(z) + c_4\mathcal{F}(z) = 0,
\end{aligned} \tag{4.21}$$

where

$$c_1 = -2 - 2a_{R_1}b - a_{R_2}b + 3b^2, \tag{4.22}$$

$$b_1 = 4 + 2a_{R_1}b + a_{R_2}b - 9b^2, \tag{4.23}$$

$$c_2 = b(a_{R_1} + a_{R_2} - 2b)(1 + a_{R_1}b - b^2), \tag{4.24}$$

$$b_2 = b(-6a_{R_1} - 4a_{R_2} + 17b - 2a_{R_1}^2 b - 2a_{R_1}a_{R_2}b + 14a_{R_1}b^2 + 6a_{R_2}b^2 - 21b^3), \quad (4.25)$$

$$a_2 = b(5a_{R_1} + 3a_{R_2} - 19b + a_{R_1}^2 b + a_{R_1}a_{R_2}b - 11a_{R_1}b^2 - 5a_{R_2}b^2 + 27b^3), \quad (4.26)$$

$$c_3 = b(1 - b^2)(2a_{R_1} - 3b - 2a_{R_1}^2 b - 2a_{R_1}a_{R_2}b + 10a_{R_1}b^2 + 4a_{R_2}b^2 - 11b^3), \quad (4.27)$$

$$b_3 = b(1 - b^2)(-2a_{R_1} + 8b + 2a_{R_1}^2 b + 2a_{R_1}a_{R_2}b - 16a_{R_1}b^2 - 7a_{R_2}b^2 + 31b^3), \quad (4.28)$$

$$c_4 = (-1 + b)b^2(1 + b)(1 - a_{R_1}b + 3b^2)(2 - a_{R_1}b - a_{R_2}b + 4b^2). \quad (4.29)$$

#### 4.1 Okubo theory: classification of Fuchsian systems

Our goal here is to characterise, in a clear way, the fourth-order Fuchsian equation (4.21). By comparing this equation to the one obtained in [4] for the 4-point conformal block with a semi-degenerate field, we want to clarify the consequences of replacing a field in the fundamental representation with a field in the adjoint one. In [24, 25] Okubo developed a theory to study the global properties of Fuchsian systems. He showed first that any Fuchsian equation can be put in a form of a Fuchsian system called the Okubo normal form. This form is particularly convenient for the definition of systems that are ‘*accessory-free*’, which are the systems that are uniquely determined, up to trivial transformations, by the positions of the singularities and by the differences between the exponents associated to each singularity. We refer the reader to [26], or to Chapter 20 of [27] for a precise definition of the accessory parameters. For accessory-free systems, there is an algorithm to determine the corresponding monodromy group. An introduction to the global properties of Fuchsian systems, and to Okubo theory, can be found in [12].

Consider a function  $F(z)$  that obeys an order- $n$  Fuchsian equation with  $p$  singular points  $z = \mu_i$ ,  $i = 1, \dots, p$ . The Okubo form of an order- $n$  Fuchsian equation takes the form,

$$\left( z\mathbb{I} - B \right) \frac{dX(z)}{dz} = AX(z) \quad (4.30)$$

where  $X(z)$  is an  $n$ -component vector, the components of which are functions related to  $F$  and its derivatives.  $\mathbb{I}$  is the  $\begin{pmatrix} n \times n \end{pmatrix}$  identity matrix,  $B$  is a diagonal matrix with  $p$  repeated elements,  $\mu_i$ ,  $i = 1, \dots, p$ ,

$$B = \text{diag} \left( \overbrace{\mu_1, \dots, \mu_1}^{m_1}, \overbrace{\mu_2, \dots, \mu_2}^{m_2}, \dots, \overbrace{\mu_p, \dots, \mu_p}^{m_p} \right), \quad m_1 + m_2 + \dots + m_p = m, \quad (4.31)$$

and  $A$  is  $\begin{pmatrix} n \times n \end{pmatrix}$  matrix with constant entries. We assume that  $A$  is diagonalisable,

$$A \sim \text{diag} \left( \overbrace{\nu_1, \dots, \nu_1}^{n_1}, \overbrace{\nu_2, \dots, \nu_2}^{n_2}, \dots, \overbrace{\nu_q, \dots, \nu_q}^{n_q} \right) \quad (4.32)$$

The eigenvalues of the matrix  $A$  correspond to the local exponents  $\gamma_i$  associated to the singularities at infinity. Corresponding to the partition in  $B$ , the matrix  $A$  can be decomposed into  $\mu_1, \mu_2, \dots, \mu_p$  matrix blocks  $A_{ij}$ ,  $1 \leq i, j \leq p$  of size  $m_i \times m_j$ ,

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & \ddots & \ddots & \vdots \\ A_{p1} & \cdots & \cdots & A_{pp} \end{pmatrix}, \quad (4.33)$$

We assume further that the diagonal blocks  $A_{ii}$ ,  $i = 1, \dots, p$  are diagonalisable,

$$A_{ii} \sim \text{diag} \left( \lambda_1^{(i)}, \dots, \lambda_{m_i}^{(i)} \right) \quad i = 1, \dots, p. \quad (4.34)$$

The eigenvalues  $\lambda_1^{(i)}$  are the local exponents associated to the singular points  $z = \mu_i$ . On the assumptions (4.32) and (4.34) [28], an irreducible Fuchsian system (4.30) is accessory-free if [28],

$$\mathcal{N} \equiv n^2 - n + 2 - \sum_{i=1}^p m_i^2 - \sum_{j=1}^q n_j^2 = 0, \quad (4.35)$$

In Okubo theory, the Fuchsian systems are then characterized by the set of integers  $m_i$  and  $n_i$  describing the multiplicities of the eigenvalues of the matrices  $B$  and  $A$ . Consider an order- $n$  Fuchsian equation. There are eight classes of systems which are known to be accessory-free,

- I:  $m_1 = m - 1, m_2 = 1$  and  $n_1 = n_2 = \dots = n_n = 1$   
I\*:  $m_1 = m_2 = \dots = m_n = 1$  and  $n_1 = n - 1, n_2 = 1$
- ( $n = 2l, l \geq 2$ )  
II:  $m_1 = m_2 = l$  and  $n_1 = l, n_2 = l - 1, n_3 = 1$   
II\*:  $m_1 = l, m_2 = l - 1, m_3 = 1$  and  $n_1 = n_2 = l$
- ( $n = 2l + 1, l \geq 2$ )  
III:  $m_1 = l + 1, m_2 = l$  and  $n_1 = l, n_2 = l, n_3 = 1$   
III\*:  $m_1 = l, m_2 = l, m_3 = 1$  and  $n_1 = l + 1, n_2 = l$
- ( $n = 6$ )  
IV:  $m_1 = 4, m_2 = 2$  and  $n_1 = n_2 = n_3 = 2$   
IV\*:  $m_1 = m_2 = m_3 = 2$  and  $n_1 = 4, n_2 = 2$

Class-I corresponds to the generalized hypergeometric equations. These are the equations obtained in [4], or more recently in [11]. The differential equations obeyed by  $n + 2$ -point

$\mathcal{W}_N$  conformal blocks involving  $n_1$  fundamental and  $n_2$  anti-fundamental fully-degenerate fields, with  $n_1 + n_2 = n$ , are related to the Calogero-Sutherland Hamiltonian [21, 29].

### Okubo form of the fourth-order differential equation

The differential equation (4.21) is a Fuchsian differential equation with coefficients  $a_i, b_i$  and  $c_i$  that are functions of  $a_{R_1}, a_{R_2}$  and  $b$ . In our case, we have

$$B = \text{diag} \left( 1, 1, 0, 0 \right), \quad \text{and} \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad (4.36)$$

where

$$A_{11} = \begin{pmatrix} 0 & 1 \\ -2 \left( 1 + 2b^2 \right)^2 & 6b^2 + 3 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (4.37)$$

$$\begin{aligned} \left( A_{21} \right)_{11} &= (b + b^3) \left( 5(2a_{R_1} + a_{R_2})b^2 + 2(2a_{R_1} + a_{R_2}) - 4b^3 - 2b \right), \\ \left( A_{21} \right)_{12} &= -(b + b^3) \left( 4a_{R_1} + 2a_{R_2} - b \right), \\ \left( A_{21} \right)_{21} &= -(b + b^3) \left( a_{R_1}^2(6b + 13b^3) + (a_{R_2} - 2b) \left( a_{R_2}(2 + 5b^2) - 2(b + 2b^3) \right) b \right) \\ &\quad - (b + b^3) \left( a_{R_1} \left( 2 + a_{R_2}(6b + 13b^3) - 3(b^2 + 7b^4) \right) \right), \\ \left( A_{21} \right)_{22} &= (b + b^3) \left( 4a_{R_1}^2 b + a_{R_1}(4a_{R_2}b - 6b^2 + 4) + (2a_{R_2} - b)(a_{R_2} - 2b)b \right), \end{aligned} \quad (4.38)$$

$$A_{22} = \begin{pmatrix} 0 & 1 \\ -(a_{R_1} + a_{R_2} - 2b) \left( 1 + (a_{R_1} - b)b \right) b & -1 - (2a_{R_1} + a_{R_2})b + 3b^2 \end{pmatrix}, \quad (4.39)$$

The matrix  $A$  can be diagonalized,

$$A \sim \text{diag} \left( 2 - a_{R_1}b - a_{R_2}b + 4b^2, 1 - a_{R_1}b + 3b^2, b^2, -1 + b^2 \right) \quad (4.40)$$

One can check that these eigenvalues, consistently with (4.20), correspond to

$$-\gamma_i - \alpha_2 - \beta_3, \quad i = 1, 2, 3 \quad (4.41)$$

and to

$$-\gamma_3 - 1 - \alpha_2 - \beta_3 \quad (4.42)$$



and are therefore associated to the singularity  $\gamma_i$  at infinity. Two of the eigenvalues differ by an integer. As we show below, we do not have logarithmic solutions, which is consistent with the fact that we are considering only semi-simple representations of the  $\mathcal{W}_3$  algebra. In this case, eigenvalues which differ by integers have to be considered as coinciding in Okubo's classification scheme<sup>3</sup>. The eigenvalues of the two  $\begin{pmatrix} 2 \times 2 \end{pmatrix}$  diagonal block matrices  $A_{11}$  and  $A_{22}$  are,

$$A_{11} \sim \text{diag} \left( 1 + 2b^2, 2(1 + 2b^2) \right), \quad \text{and} \quad A_{22} \sim \text{diag} \left( 1 + 2b^2, 2(1 + 2b^2) \right) \quad (4.43)$$

are associated respectively to the  $\beta_i$  and  $\alpha_i$  local exponents,  $i = 1, 2$ . In Okubo's classification scheme, our Fuchsian differential equation corresponds to

$$n = 4, \quad q = 2, \quad m_1 = m_2 = 2, \quad p = 2, \quad n_1 = 2 \quad n_2 = 1 \quad (4.44)$$

In other words, equation (4.21) belongs to class-II of accessory-free Fuchsian systems. This result is interesting, as one would have expected that the presence of multiplicities in the adjoint representation implies accessory parameters in the Fuchsian equation. For instance in [4] a Fuchsian equation with accessory parameters was found in the so called '*heavy*' semi-classical limit of the 3-point Toda correlation function. Finally, it is interesting to notice that in [9]  $n$ -point conformal block,  $n > 4$ , with fully-degenerate fields were explicitly given in terms of  ${}_3F_2$  hypergeometric functions, that is class-I. It would be interesting to check if, in an appropriate limit, one could recover the solutions of the class-II Fuchsian system obtained in our case.

## 5 Constructing monodromy-invariant function

Because we chose the operators such that we have admissible fusion, as implied in particular in the relation (4.11), the conformal blocks have the following Coulomb gas representation,

$$\left\langle \left( \oint du V_{b\vec{e}_1}(u) \right) \left( \oint dv V_{b\vec{e}_2}(v) \right) V_{\vec{\alpha}_R}(0) V_{-b\vec{\omega}_1}(z) V_{-b(\vec{\omega}_1+\vec{\omega}_2)}(1) V_{2Q\vec{\rho}-\vec{\alpha}_R+b\vec{\omega}_1}(\infty) \right\rangle, \quad (5.1)$$

One can directly verify that the charges of the vertices entering the above formula add up to  $2Q\vec{\rho}$ , thus satisfying the  $\mathfrak{sl}_3$  Coulomb gas neutrality condition (2.49) with  $n_1 = n_2 = 1$  and  $m_1 = m_2 = 0$ . The above integral yields

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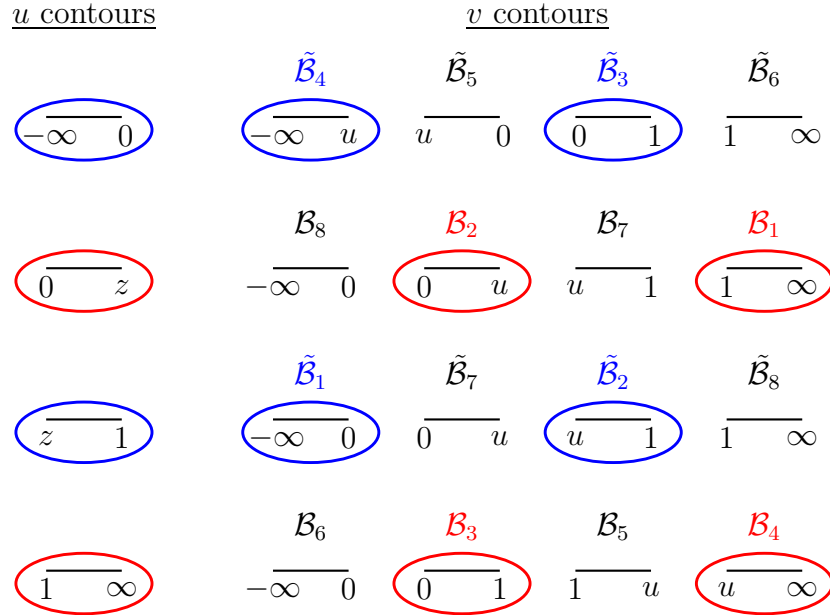
<sup>3</sup> We thank Professor Y. Haraoka for explaining this point to us.

$$z^{b\vec{\omega}_1 \cdot \vec{\alpha}_R} (1-z)^{b(\vec{\omega}_1 + \vec{\omega}_2) \cdot \vec{\alpha}_R} \oint du \oint dv \left( u^{-b\vec{e}_1 \cdot \vec{\alpha}_R} v^{-b\vec{e}_2 \cdot \vec{\alpha}_R} (u-z)^{b^2} (u-1)^{b^2} (v-1)^{b^2} (v-u)^{b^2} \right) \quad (5.2)$$

We need to look for all independent ways of positioning the integration contours. Given that our differential equation is fourth-order, we expect four linearly-independent such choices. It is convenient to use contours that are open paths between singularities. These integrals can be obtained by shrinking the closed paths that surround these singularities. In particular, each variable  $u$  and  $v$  is integrated along one of the four paths,

$$\mathcal{C}_1 : -\infty \rightarrow 0, \quad \mathcal{C}_2 : 0 \rightarrow z, \quad \mathcal{C}_3 : z \rightarrow 1, \quad \mathcal{C}_4 : 1 \rightarrow \infty \quad (5.3)$$

We need to be careful, as the integration paths of  $u$  and  $v$  are not allowed to intersect, and we need to specify which contour lies on top of the other contour. There are 16 possible choices, which are illustrated in the following Figure,



In the above Figure, a function  $\mathcal{B}_i$  or  $\tilde{\mathcal{B}}_i$ ,  $i = 1, 2, 3, 4$ , is associated to each of the sixteen contour choices. For any of the four domains of integration of the variable  $u$ , indicated on the left column, there correspond four possible domains of integration of the variable  $v$ , represented on the same row on the right. For instance the  $\mathcal{B}_5$  function is given by the integral where  $u \in [1, \infty]$  and  $v \in [1, u]$ ,

$$\mathcal{B}_5(z) = z^{b\vec{\omega}_1 \cdot \vec{\alpha}_R} \left(1 - z\right)^{b(\vec{\omega}_1 + \vec{\omega}_2) \cdot \vec{\alpha}_R} \int_1^\infty du \int_1^u dv \left( u^{-b\vec{e}_1 \cdot \vec{\alpha}_R} v^{-b\vec{e}_2 \cdot \vec{\alpha}_R} (z - u)^{b^2} (1 - u)^{b^2} (v - 1)^{b^2} (v - u)^{b^2} \right), \quad (5.4)$$

The red (blue) ellipses indicate the  $s$ - ( $t$ -) channel solution basis of the differential equation (4.21). Let us consider the  $s$ -channel that correspond to the conformal blocks with abelian monodromy around  $z = 0$ . They have the form  $\mathcal{B}_i(z) = z^{\alpha_i} \sum_{n \geq 0} a_n^i z^n$ . Consider for instance the  $\mathcal{B}_1(z)$  integral,

$$\mathcal{B}_1(z) = z^{b\vec{\omega}_1 \cdot \vec{\alpha}_R} \left(1 - z\right)^{b(\vec{\omega}_1 + \vec{\omega}_2) \cdot \vec{\alpha}_R} \int_0^z du \int_1^\infty dv \left( u^{-b\vec{e}_1 \cdot \vec{\alpha}_R} v^{-b\vec{e}_2 \cdot \vec{\alpha}_R} (z - u)^{b^2} (1 - u)^{b^2} (v - 1)^{b^2} (v - u)^{b^2} \right), \quad (5.5)$$

By a simple change of variables, the  $\mathcal{B}_1(z)$  can be written as,

$$\mathcal{B}_1(z) = z^{\alpha_1} \left(1 - z\right)^{b(\vec{\omega}_1 + \vec{\omega}_2) \cdot \vec{\alpha}_R} \int_0^1 du \int_1^\infty dv \left( u^{-b\vec{e}_1 \cdot \vec{\alpha}_R} v^{-b\vec{e}_2 \cdot \vec{\alpha}_R} (1 - u)^{b^2} (1 - zu)^{b^2} (v - 1)^{b^2} (v - zu)^{b^2} \right), \quad (5.6)$$

that has the correct small  $z$  behavior. The fact that  $\mathcal{B}_i$  has a simple monodromy around  $z = 0$  can be understood by contour deformation techniques. We have verified that  $\mathcal{B}_1(z)$  and  $\mathcal{B}_2(z)$  are solutions of the differential equation (4.21). Finally, we have also verified that the functions  $\mathcal{B}_i(z)$ ,  $i = 3, 4, 5, 6$ , are all solutions of the differential equation with local exponents  $\alpha_3$ , given in (4.18). For instance, it is easy to verify that,

$$\mathcal{B}_3(z) = z^{b\vec{\omega}_1 \cdot \vec{\alpha}_R} \left(1 - z\right)^{b(\vec{\omega}_1 + \vec{\omega}_2) \cdot \vec{\alpha}_R} \int_1^\infty du \int_0^1 dv \left( u^{-b\vec{e}_1 \cdot \vec{\alpha}_R} v^{-b\vec{e}_2 \cdot \vec{\alpha}_R} (u - z)^{b^2} (u - 1)^{b^2} (1 - v)^{b^2} (u - v)^{b^2} \right), \quad (5.7)$$

behaves for small  $z$  as  $z^{\alpha_3}(a_0 + a_1 z + \dots)$ . As we discuss below, only two functions among the  $\mathcal{B}_i(z)$ ,  $i = 3, 4, 5, 6$  are independent. We choose arbitrary  $\mathcal{B}_3(z)$  and  $\mathcal{B}_4(z)$  as the other two functions of the  $s$ -channel basis. The same consideration can be done for the  $t$ -channel basis, associated to the  $\beta_i$  local exponents, defined in (4.18).

## 5.1 The monodromy group

The monodromy group is generated by the monodromy matrices  $\mathcal{M}_0$  and  $\mathcal{M}_1$ . In the reference  $s$ -channel basis

$$\{\mathcal{B}_1(z), \mathcal{B}_2(z), \mathcal{B}_3(z), \mathcal{B}_4(z)\} \quad (5.8)$$

the matrix  $\mathcal{M}_0$  is diagonal:

$$\mathcal{M}_0 = \text{diag} \left( e^{2\pi i \alpha_1}, e^{2\pi i \alpha_2}, e^{2\pi i \alpha_3}, e^{2\pi i \alpha_3} \right) \quad (5.9)$$

To find the matrix  $\mathcal{M}_1$ , we need the  $4 \times 4$  matrix  $L_{i,j}$  which relates the  $\mathcal{B}_i$  to the  $\tilde{\mathcal{B}}_j$ ,

$$\tilde{\mathcal{B}}_j = L_{j,i} \mathcal{B}_i \quad (5.10)$$

Noting that

$$\tilde{\mathcal{M}}_1 = \text{diag} \left( e^{2\pi i \alpha_1}, e^{2\pi i \alpha_2}, e^{2\pi i \alpha_3}, e^{2\pi i \alpha_3} \right) \quad (5.11)$$

we have,

$$\mathcal{M}_1 = L^{-1} \tilde{\mathcal{M}}_1 L \quad (5.12)$$

We could find the matrix  $L$  explicitly by applying the residue theorem systematically. For instance, fix the variable  $u \in (0, z)$  as a parameter and consider the  $v$  integrals. We denote

$$x = -b \vec{e}_1 \cdot \vec{\alpha}_R, \quad y = -b \vec{e}_2 \cdot \vec{\alpha}_R. \quad (5.13)$$

Fixing  $u$  and moving the variable  $v$ , Cauchy theorem provides relations of the kind

$$\begin{aligned} \left( \int_{-\infty}^0 dv \dots \right) + e^{\pm i \pi y} \left( \int_0^u dv \dots \right) + e^{\pm i \pi (y+b^2)} \left( \int_u^1 dv \dots \right) \\ + e^{\pm i \pi (y+2b^2)} \left( \int_1^\infty dv \dots \right) = 0. \end{aligned} \quad (5.14)$$

Using these, one can find a linear relation between the contour integrals, such as

$$\sin \left( \pi b^2 \right) \mathcal{B}_2(z) + \sin \left( \pi (y + b^2) \right) \mathcal{B}_7(z) + \sin \left( \pi (y + 2b^2) \right) \mathcal{B}_1(z) = 0. \quad (5.15)$$

Similarly, by fixing  $v$  and moving the variable  $u$ , we obtain relation between  $\mathcal{B}_i$  and  $\tilde{\mathcal{B}}_j$  functions, such as

$$\sin \left( \pi b^2 \right) \tilde{\mathcal{B}}_4(z) + \sin \left( \pi(x + b^2) \right) \mathcal{B}_8(z) + \sin \left( \pi(x + 2b^2) \right) \tilde{\mathcal{B}}_1(z) \\ + \sin \left( \pi(x + 3b^2) \right) \mathcal{B}_6(z) = 0 \quad (5.16)$$

We find out 12 independent relations which allow to write the matrix  $L$  (using short notation  $[x] = \sin(\pi x)$ ) as follows

$$L = \begin{pmatrix} -\frac{[b^2][x]}{[b^2+x][b^2+y]} & \frac{[b^2][b^2+x+y]}{[b^2+y][2b^2+x+y]} & \frac{[b^2][3b^2+x][b^2+x+y]}{[b^2+x][b^2+y][2b^2+x+y]} & -\frac{[b^2][x][4b^2+x+y]}{[b^2+x][b^2+y][2b^2+x+y]} \\ \frac{[x][2b^2+y]}{[b^2+x][b^2+y]} & \frac{[y][b^2+x+y]}{[b^2+y][2b^2+x+y]} & -\frac{[3b^2+x][y][3b^2+x+y]}{[b^2+x][b^2+y][2b^2+x+y]} & -\frac{[2b^2+x][2b^2+y][4b^2+x+y]}{[b^2+x][b^2+y][2b^2+x+y]} \\ \frac{[b^2][2b^2+y]}{[b^2+x][b^2+y]} & -\frac{[b^2]^2}{[b^2+y][2b^2+x+y]} & \frac{[b^2+y][b^2+x+y]+[y][4b^2+x+y]}{[b^2+y][b^2+x+y][2b^2+x+y]} & \frac{[b^2][2b^2+y][4b^2+x+y]}{[b^2+x][b^2+y][2b^2+x+y]} \\ -\frac{[b^2]}{[b^2+x][b^2+y]} & -\frac{[b^2][y]}{[b^2+y][2b^2+x+y]} & -\frac{[b^2][3b^2+x][y]}{[b^2+x][b^2+y][2b^2+x+y]} & -\frac{[x][b^2+y]+[3b^2+x][2b^2+y]}{[b^2+y][b^2+x+y][2b^2+x+y]} \end{pmatrix}. \quad (5.17)$$

## 5.2 Monodromy-invariant correlation function

We consider the general 4-point correlation function

$$G(z, \bar{z}) = \mathcal{B}(z)_T X \mathcal{B}(\bar{z}), \quad (5.18)$$

where  $\mathcal{B}(z) = (\mathcal{B}_1(z), \mathcal{B}_2(z), \mathcal{B}_3(z), \mathcal{B}_4(z))$  is the array of  $s$ -channel solutions,  $\mathcal{B}(z)_T$  its transpose and  $X$  is a  $4 \times 4$  matrix of coefficients which do not depend on  $z$ . We look for constraints on  $X_{i,j}$ , imposed by demanding

$$\mathcal{M}_0 G(z, \bar{z}) = G(z, \bar{z}) \quad \mathcal{M}_1 G(z, \bar{z}) = G(z, \bar{z}) \quad (5.19)$$

From the first relation in (5.19) and from (5.9), the matrix  $X$  has to have the following form

$$X = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & x_5 + i x_6 \\ 0 & 0 & x_5 - i x_6 & x_4 \end{pmatrix}, \quad (5.20)$$

Up to a global normalization, we have five unknown constants  $x_i/x_1$ ,  $i = 2, \dots, 6$  to determine. This will be fixed by imposing the second constraint in (5.19). In order to know the action of  $\mathcal{M}_1$  on  $G$ , we write the function  $G$  in terms of the  $t$ -channel basis  $\tilde{\mathcal{B}}(z) = (\tilde{\mathcal{B}}_1(z), \tilde{\mathcal{B}}_2(z), \tilde{\mathcal{B}}_3(z), \tilde{\mathcal{B}}_4(z))$

$$G(z, \bar{z}) = \tilde{\mathcal{B}}(z)_T Y \tilde{\mathcal{B}}(\bar{z}), \quad (5.21)$$

In term of the monodromy matrix  $L$ , given in (5.17), the matrix  $Y$  is

$$Y = \left( L^{-1} \right)^T X L^{-1} \quad (5.22)$$

Taking (5.11) into account, the relation  $\mathcal{M}_1 G(z, \bar{z}) = G(z, \bar{z})$  implies five equations:

$$Y_{1,2} = 0, \quad Y_{1,3} = 0, \quad Y_{1,4} = 0, \quad Y_{2,3} = 0, \quad Y_{2,4} = 0. \quad (5.23)$$

The above equation form a system of linear equations in the coefficients  $x_i$ . We have checked that the five relations are independent. The system (5.23) has rank five and allows to fix the five ratios  $x_i/x_1$ ,  $i = 2, \dots, 6$ . We note that  $x_i^{\text{sol}}$ ,  $i = 1, \dots, 6$  represent the solutions of the above system. The value  $x_1^{\text{sol}}$  can be chosen arbitrary to fix the global normalization of the correlation  $G(z, \bar{z})$ . We found  $x_6^{\text{sol}} = 0$ . The other solutions  $x_i^{\text{sol}}$ ,  $i = 2, \dots, 5$ , which depend on  $x, y$  and  $b$  (or equivalently on  $a_{R_1}, a_{R_1}$  and  $b$  see (5.13)), are given in terms of trigonometric functions. It is convenient to express  $G(z, \bar{z})$  in terms of the normalized functions,

$$\mathcal{B}_i^{\text{norm}}(z) = \frac{\mathcal{B}_i(z)}{\mathcal{B}_i(0)}, \quad (5.24)$$

where:

$$\mathcal{B}_1(0) = \frac{\Gamma(1+b^2)^2 \Gamma(1+x) \Gamma(-1-2b^2-y)}{\Gamma(2+b^2+x) \Gamma(-b^2-y)}, \quad (5.25)$$

$$\mathcal{B}_2(0) = \frac{\Gamma(1+b^2)^2 \Gamma(1+y) \Gamma(2+b^2+x+y)}{\Gamma(2+b^2+y) \Gamma(3+2b^2+x+y)}, \quad (5.26)$$

$$\begin{aligned} \mathcal{B}_3(0) &= \frac{\Gamma(1+b^2)^2 \Gamma(-1-3b^2-x) \Gamma(1+y)}{\Gamma(-2b^2-x) \Gamma(2+b^2+y)} \times \\ &\times {}_3F_2(-b^2, -1-3b^2-x, 1+y; -2b^2-x, 2+b^2+y|1), \end{aligned} \quad (5.27)$$

$$\begin{aligned} \mathcal{B}_4(0) &= \frac{\Gamma(1+b^2)^2 \Gamma(-1-2b^2-y) \Gamma(-2-4b^2-x-y)}{\Gamma(-b^2-y) \Gamma(-1-3b^2-x-y)} \times \\ &\times {}_3F_2(-b^2, -1-2b^2-y, -2-4b^2-x-y; -b^2-y, -1-3b^2-x-y|1). \end{aligned} \quad (5.28)$$

Setting  $x_1^{\text{sol}} = \mathcal{B}_1(0)^2$  and

$$X_i = \frac{\mathcal{B}_i(0)^2}{\mathcal{B}_1(0)^2} x_i^{\text{sol}}, \quad (5.29)$$

the monodromy invariant correlation function takes the form,

$$G(z, \bar{z}) = \mathcal{B}_1^{\text{norm}}(z)\mathcal{B}_1^{\text{norm}}(\bar{z}) + X_2\mathcal{B}_2^{\text{norm}}(z)\mathcal{B}_2^{\text{norm}}(\bar{z}) + X_3\mathcal{B}_3^{\text{norm}}(z)\mathcal{B}_3^{\text{norm}}(\bar{z}) \\ + X_4\mathcal{B}_4^{\text{norm}}(z)\mathcal{B}_4^{\text{norm}}(\bar{z}) + X_5 \left( \mathcal{B}_3^{\text{norm}}(z)\mathcal{B}_4^{\text{norm}}(\bar{z}) + \mathcal{B}_4^{\text{norm}}(z)\mathcal{B}_3^{\text{norm}}(\bar{z}) \right). \quad (5.30)$$

The function  $G(z, \bar{z})$  have therefore the following small  $z$  behavior,

$$G(z, \bar{z}) = |z|^{2\alpha_1} \left( 1 + \mathcal{O}(z, \bar{z}) \right) + X_2 |z|^{2\alpha_2} \left( 1 + \mathcal{O}(z, \bar{z}) \right) \\ + (X_3 + 2X_5 + X_4) |z|^{-2\alpha_3} \left( 1 + \mathcal{O}(z, \bar{z}) \right), \quad (5.31)$$

where the exponents have been defined in (4.18) and correspond to the channels (4.13), (4.14) and (4.15). We can now directly compare our findings with the structure constants computed in [4] and re-derived here in D. In the second channel, we found that,

$$X_2 = \frac{\gamma(-x)\gamma(2+x+b^2)\gamma(-y-b^2)\gamma(2+y+2b^2)}{\gamma(-y)\gamma(2+y+b^2)\gamma(-1-x-y-b^2)\gamma(3+x+y+2b^2)}. \quad (5.32)$$

Noting as  $\vec{\alpha}_M^{(1)}$  and  $\vec{\alpha}_M^{(2)}$  the charges  $\vec{\alpha}_M$  corresponding respectively to (4.13) and to (4.14), one can directly verify that,

$$X_2 = \frac{C_{-b\vec{\omega}_1, \vec{\alpha}_R}^{\vec{\alpha}_M^{(2)}} C_{-b\vec{\rho}, \vec{\alpha}_M^{(2)}}^{\vec{\alpha}_L}}{C_{-b\vec{\omega}_1, \vec{\alpha}_R}^{\vec{\alpha}_M^{(1)}} C_{-b\vec{\rho}, \vec{\alpha}_M^{(1)}}^{\vec{\alpha}_L}}, \quad (5.33)$$

where we recall that the  $\vec{\alpha}_L$  is related to  $\vec{\alpha}_R$  by the (4.11). The structure constants  $C_{-b\vec{\omega}_1, \vec{\alpha}_R}^{\vec{\alpha}_M^{(i)}}$  and  $C_{-b\vec{\rho}, \vec{\alpha}_M^{(i)}}^{\vec{\alpha}_L}$  are given in (D.5) and in (D.8). Note that the normalization of the fields  $\Phi_{\vec{\alpha}_L}$  and  $\Phi_{\vec{\alpha}_R}$  are irrelevant in this ratio, and therefore we can use the vertex operators to compute these three point functions.

More interesting is the channel (4.15),  $\vec{\alpha}_M = \vec{\alpha}_L$  associated with the constants  $X_3$ ,  $X_4$  and  $X_5$ . We recall that the existence of a two-dimensional space of solutions spanned by  $\mathcal{B}_3(z)$  and  $\mathcal{B}_4(z)$  is related to the two-fold multiplicity of the 0 weight in the adjoint representation. In the OPE between  $\Phi_{2211}$  and  $\Phi_L$  this ambiguity can be expressed by a free parameter  $\iota$  appearing from the first-order of the expansion,

$$\Phi_{2211}(z)\Phi_L(0) = \\ z^{-h}\Phi_L(0) + z^{-h+1} \left( \frac{h}{2h_L}L_{-1} + \iota(-3q_LL_{-1} + 2h_LW_{-1}) \right) \Phi_L(0) + \mathcal{O}(z^{-h+2}). \quad (5.34)$$

Any conformal block obtained by the linear combination  $\mathcal{B}_3(z) + s(\iota)\mathcal{B}_4(z)$ , which is

therefore also a solution of the differential equation (4.21), is associated to a particular value of  $\iota$ . Note that in the above expansion we have chosen a particular basis of state at the first level, namely  $L_{-1}\Phi_L$  and  $\chi(z) \equiv (-3q_L L_{-1} + 2h_L W_{-1})\Phi_L(z)$ . This choice can be considered as the most natural as  $\chi(z)$  is a *Virasoro* primary field (it is  $\mathcal{W}_3$  primary only if  $\Phi_L$  is associated to the  $\vec{\omega}_1$  and  $\vec{\omega}_2$  representation). We have verified that

$$X_3 + 2X_5 + X_4 = \frac{C_{-b\vec{\omega}_1, \vec{\alpha}_R}^{\vec{\alpha}_L} C_{-b\vec{\rho}, \vec{\alpha}_L}^{\vec{\alpha}_L}}{C_{-b\vec{\omega}_1, \vec{\alpha}_R}^{\vec{\alpha}_M^{(1)}} C_{-b\vec{\rho}, \vec{\alpha}_M^{(1)}}^{\vec{\alpha}_L}}, \quad (5.35)$$

where the structure constant  $C_{-b\vec{\rho}, \vec{\alpha}_L}^{\vec{\alpha}_L}$  has been derived in (D.16).

The above result shows that the monodromy requirement eliminates the ambiguities due to the presence of multiplicities. This is of course expected since  $G(z, \bar{z})$  has a Coulomb gas representation of type (2.47), which has a simple monodromy by construction. On the other hand, the Coulomb gas approach of type (2.50) shows in the most transparent way how the monodromy solves the problem of ambiguities. Moreover, we have computed all the constants  $X_i$  in (5.29) from the solutions of the linear system (5.23) and from the values of the norms (5.29). We stress that these constants are hardly accessible *via* the study of the Coulomb gas representation (2.47) of  $G(z, \bar{z})$ , while the direct computation of the 3-point function *via* the Coulomb gas gives access only to a particular combination of these three values.

## 6 Summary and discussion

In this paper we studied  $\mathcal{W}_3$  Toda conformal field theory. We considered the explicit construction of matrix elements, conformal blocks and correlation functions of primary fields in the case where the primary fields do not belong to the special class that is available to the AGT approach [30, 31]. We recall here that, in general,  $\mathcal{W}_N$  AGT allows one to construct matrix elements for the fields with highest-weights proportional either  $\omega_1$  or  $\omega_{N-1}$  fundamental weights of  $\mathfrak{sl}_N$ . In this case, the correspondence between 2-dimensional conformal field theory and 4-dimensional supersymmetric gauge theories, as proposed in [30], is available, and  $\mathcal{W}_N$  conformal blocks are equal to Nekrasov instanton partition functions [31]. The core of the problem of extending these results is that for  $N \geq 3$  the general matrix element of a primary field between two descendant states is not expressed solely in terms of the primary 3-point function but involves also an infinite set of new independent basic matrix elements, so to say,  $\mathcal{W}_N$ -partners. For the special class of semi-degenerate fields this problem can be resolved and all matrix elements, as well as the coefficients of the operator product expansion, can be fixed uniquely in terms of 3-point functions of primary fields<sup>4</sup>.

In this paper, we focused on the simplest example of non-AGT type: a primary fully-degenerate field in the adjoint representation of  $\mathfrak{sl}_3$  algebra in  $\mathcal{W}_3$  Toda theory. What we

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<sup>4</sup> See [7, 9, 10, 32, 33] for recent works towards a more general analysis.



call non-AGT-type refers strictly to the case in which the linear identities of type (3.12) are not sufficient, and one needs the non-linear identity of type (3.21). We showed how null-state conditions in this case allow one to determine the operator product expansion, and, in particular, its first level coefficients, which, as one might think naively, could remain unfixed. This case is of particular interest since it contains non-trivial effects of the multiplicity problem. Using representation theory, we showed that these matrix elements can be computed explicitly.

Next, we constructed 4-point conformal blocks which involves matrix elements of the new type. We checked our results against the differential equation which follows from the null-state conditions for one semi-degenerate fundamental and one fully-degenerate adjoint fields. We found the Okubo form of the obtained fourth-order Fuchsian differential equation, and showed that it is an accessory-free Fuchsian system of the class-II. We solved the differential equation in terms of Coulomb gas integrals and computed the monodromy group. Finally, we showed that in the presence of multiplicity at the level of the matrix elements, using the bootstrap technique, the correlation functions can be constructed uniquely and do not contain arbitrary parameters. As a byproduct, we obtained explicit expressions for the particular structure constants that are not accessible by means of the Coulomb gas representation.

A natural question for further study is the generalisation of the discussed methods for more general classes of non-fundamental or anti-fundamental fields. In this context, constructing semi-degenerate fields in higher representations of  $\mathfrak{sl}_N$  seems to be relevant. Work on this problem is in progress [34]. Another interesting question is the possible extension of rational and non-rational versions of  $\mathcal{W}_N$  AGT [22, 31, 35, 36] on the class of non-fundamental fields. This problem requires constructing vertex operators in the composite  $\mathcal{W}_N \times \mathcal{H}$  conformal field theory, where  $\mathcal{H}$  is the Heisenberg algebra. In this context, a search for the *dressing* Heisenberg field for the non-fundamental fields  $\mathcal{W}_N$  is required.

In [32, 33], general  $\mathcal{W}_N$  3-point functions are computed, starting from topological string partition functions. Strictly speaking, these results are in the context of a  $q$ -deformed version of a  $\mathcal{W}_N \times \mathcal{H}$  conformal field theory, where  $\mathcal{H}$  is the Heisenberg algebra. The  $q \rightarrow 1$  limit of these results is, at this stage, not entirely straightforward. In [10], an approach towards  $q$ -deformed  $\mathcal{W}_N \times \mathcal{H}$   $(n+2)$ -point conformal blocks that involve  $\mathcal{W}_N$  primary fields, in representations of  $\mathfrak{sl}_N$  that are higher than the fundamental, was briefly proposed. The basic idea in this proposal is to take suitable limits of conformal blocks of primary fields in the fundamental of  $\mathfrak{sl}_N$ . It remains to carry out this proposal in detail.

## A Shapovalov matrices, at level-one and level-two

We give in the following the Shapovalov matrices, at level-one and level-two, associated to the primary field  $\Phi_M$  characterized by the  $h_M, q_M$  quantum numbers. We use here the notation introduced in 2.11.

### The level-one block

There are two states at level-one,  $I_1 = \{1; \emptyset\}$  and  $I_2 = \{\emptyset; 1\}$ . In the basis  $(I_1, I_2)$ , the matrix  $H_1$  is

$$H_1 = \begin{pmatrix} 2h_M & 3q_M \\ 3q_M & \frac{2}{3} \left( h_M^2 + \frac{h_M}{5} \right) - \frac{1}{240}(5c + 22)h_M \end{pmatrix} \quad (\text{A.1})$$

### The level-two block

There are five states at level-two.  $I_1 = \{2; \emptyset\}$ ,  $I_2 = \{1, 1; \emptyset\}$ ,  $I_3 = \{\emptyset; 2\}$ ,  $I_4 = \{\emptyset; 1, 1\}$ , and  $I_5 = \{1; 1\}$ . In the basis  $(I_1, I_2, I_3, I_4, I_5)$ , we have the  $5 \times 5$  matrix

$$H_2 = \begin{pmatrix} \frac{c}{2} + 4\Delta_M & 6h_M & 6q_M & 6q_M & 6q_M \\ 6\Delta_M & 4h_M(2h_M + 1) & 12q_M & 12q_M & 12q_M \\ 6q_M & 12q_M & \frac{4}{3}h_M^2 + (c+6)\frac{h_M}{6} & \frac{4}{3}h_M^2 + (c+6)\frac{h_M}{6} & \frac{4}{3}h_M^2 + (c+6)\frac{h_M}{6} \\ \left( 32h_M + 2 - c \right) \frac{5h_M}{48} & 18q_M^2 + 4h_M^2 + (2-c)\frac{h_M}{8} & \left( c + 14 \right) \frac{q_M}{8} + 6h_M q_M & \left( c + 14 \right) \frac{q_M}{8} + 6h_M q_M & \left( c + 14 \right) \frac{q_M}{8} + 6h_M q_M \\ 9q_M & 12h_M q_M + 6q_M & \left( -c + 32h_M + 2 \right) \frac{h_M}{12} & \left( -c + 32h_M + 2 \right) \frac{h_M}{12} & \left( -c + 32h_M + 2 \right) \frac{h_M}{12} \\ -\frac{5}{48} \left( c - 32h_M - 2 \right) h_M & h_M & 9q_M & 9q_M & 9q_M \\ 18q_M^2 + 4h_M^2 + (2-c)\frac{h_M}{8} & \frac{h_M}{8} & 12h_M q_M + 6q_M & 12h_M q_M + 6q_M & 12h_M q_M + 6q_M \\ \left( c + 14 \right) \frac{q_M}{8} + 6h_M q_M & \left( 32h_M + 2 - c \right) \frac{h_M}{12} & \left( 32h_M + 2 - c \right) \frac{h_M}{12} & \left( 32h_M + 2 - c \right) \frac{h_M}{12} & \left( 32h_M + 2 - c \right) \frac{h_M}{12} \\ \left( c^2 - 52c + 612 \right) \frac{h_M^2}{1152} + \left( c^2 - 36c + 68 \right) \frac{h_M}{2304} + \frac{16h_M^4 + (18-c)\Delta_M^3 + 216q_M^2}{18} & \left( 32h_M + 2 - c \right) \frac{(2h_M+3)q_M}{16} & \left( 32h_M + 2 - c \right) \frac{(2h_M+3)q_M}{16} & \left( 32h_M + 2 - c \right) \frac{(2h_M+3)q_M}{16} & \left( 32h_M + 2 - c \right) \frac{(2h_M+3)q_M}{16} \\ \left( 32h_M + 2 - c \right) \frac{(2h_M+3)q_M}{16} & 9q_M^2 + \left( 32\Delta_M + 2 - c \right) \frac{(h_M+1)h_M}{24} & 9q_M^2 + \left( 32\Delta_M + 2 - c \right) \frac{(h_M+1)h_M}{24} & 9q_M^2 + \left( 32\Delta_M + 2 - c \right) \frac{(h_M+1)h_M}{24} & 9q_M^2 + \left( 32\Delta_M + 2 - c \right) \frac{(h_M+1)h_M}{24} \end{pmatrix} \quad (\text{A.2})$$

## B Matrix elements, at level-one and at level-two

We give here the expansion (3.2) till the second-order of a general  $\mathcal{W}_3$  conformal block. In the following we define

$$W_1^a \equiv \frac{\langle \Phi_M | \left( W_{-1} \Phi_1 \right) | \Phi_R \rangle}{\langle \Phi_M | \Phi_1 | \Phi_R \rangle}, \quad W_1^b \equiv \frac{\langle \Phi_L^* | \left( W_{-1} \Phi_2 \right) | \Phi_M \rangle}{\langle \Phi_L^* | \Phi_2 | \Phi_M \rangle} \quad (\text{B.1})$$

and

$$W_{1,1}^a \equiv \frac{\langle \Phi_M | \left( W_{-1}^2 \Phi_1 \right) | \Phi_R \rangle}{\langle \Phi_M | \Phi_1 | \Phi_R \rangle}, \quad W_{1,1}^b \equiv \frac{\langle \Phi_L^* | \left( W_{-1}^2 \Phi_2 \right) | \Phi_M \rangle}{\langle \Phi_L^* | \Phi_2 | \Phi_M \rangle} \quad (\text{B.2})$$

These elements are unknown for general field  $\Phi_1$  and  $\Phi_2$ .

### The level-one matrix elements

At first-order in the expansion of (3.2), we have the matrix elements

$$\Gamma'_{\{1;\emptyset\},\emptyset,\emptyset} = h_1 + h_M - h_R \quad (\text{B.3})$$

$$\Gamma'_{\{\emptyset;1\},\emptyset,\emptyset} = q_M - q_R + 2w_1 + W_1^a \quad (\text{B.4})$$

$$\Gamma_{\emptyset,\emptyset,\{1;\emptyset\}} = h_M + h_2 - h_L \quad (\text{B.5})$$

$$\Gamma_{\emptyset,\emptyset,\{\emptyset;1\}} = q_M + q_L + q_2 + W_1^b \quad (\text{B.6})$$

### The Level-two matrix elements

At the second-order in the expansion (3.2), we have,

$$\Gamma'_{\{2;\emptyset\},\emptyset,\emptyset} = 2h_1 + h_M - h_R \quad (\text{B.7})$$

$$\Gamma'_{\{1,1;\emptyset\},\emptyset,\emptyset} = (h_1 + 1 + h_M - h_R)(h_1 + h_M - h_R) \quad (\text{B.8})$$

$$\Gamma'_{\{\emptyset;2\},\emptyset,\emptyset} = q_M - w_R + 5q_1 + 2W_1^a \quad (\text{B.9})$$

$$\begin{aligned} \Gamma'_{\{\emptyset;1,1\},\emptyset,\emptyset} = & (q_M - q_R + 2q_1)(q_M - q_R + 2w_1 + 2W_1^a) + \\ & + 2(2/15 - \eta/5 + 2/3h_1)(h_M - h_1 - h_R) + \\ & + (2/15 - \eta/5 + 2/3h_1)(h_M + h_1 - h_R) + \\ & + (2/15 - \eta/5)h_1 + 2/3h_1^2 + W_{1,1}^a \end{aligned} \quad (\text{B.10})$$

$$\Gamma'_{\{1;1\},\emptyset,\emptyset} = (h_M + h_1 + 1 - h_R)(q_M - q_R + 2w_1 + W_1^a) \quad (\text{B.11})$$

$$\Gamma_{\emptyset,\emptyset,\{2;\emptyset\}} = h_M + 2h_2 - h_L \quad (\text{B.12})$$

$$\Gamma_{\emptyset,\emptyset,\{1,1;\emptyset\}} = (h_M + h_2 + 1 - h_L)(h_M + h_2 - h_L) \quad (\text{B.13})$$

$$\Gamma_{\emptyset,\emptyset,\{\emptyset;2\}} = q_M + q_L + q_2 + 2W_1^b \quad (\text{B.14})$$

$$\begin{aligned} \Gamma_{\emptyset,\emptyset,\{\emptyset;1,1\}} = & (q_M + q_L + q_2)(q_M + q_L + q_2 + 2W_1^b) \\ & + (2/15 - \eta/5 + 2/3h_2)(h_L - h_M - h_2) + \\ & + (2/15 - \eta/5 + 2/3h_M)(h_M + h_2 - h_L) + W_{1,1}^b \end{aligned} \quad (\text{B.15})$$

In the above expressions, the constant  $\eta$  has been defined in (2.8). Using the above formula and the expansion (3.2), the first two orders in the expansion of a general conformal block can be given in terms of the central charge  $c$ , of the external fields parameters  $\{h_L, q_L, h_2, q_2, h_1, q_1, h_R, q_R\}$ , of the fusion channel ones  $\{h_M, q_M\}$  and of the four elements  $W_1^{a,b}$  and  $W_{1,1}^{a,b}$ .

## C $\mathcal{W}_3$ highest-weight modules with null-states at level-two

We consider a primary field  $\Phi_{2211}$  and we want to derive the form of the two  $\mathcal{W}_3$  null-states at level-two. In general case there are five states  $(I_1, I_2, I_3, I_4, I_5)$  at level two (see A). Level-two state

$$\Psi(\lambda) = \left( c_1(\lambda)L_{-2} + c_2(\lambda)W_{-2} + c_3(\lambda)L_{-1}^2 + c_4(\lambda)L_{-1}W_{-1} + W_{-1}^2 \right) \Phi_{h,q} \quad (\text{C.1})$$

is an eigenstate of the operator  $W_0$  of eigenvalue  $\lambda + q$ , if

$$c_1(\lambda) = \frac{1}{128\eta^2} \left( \lambda^4 - (8f + 16\eta^2)\lambda^2 + 64\eta^4 \right), \quad (\text{C.2})$$

$$c_2(\lambda) = \frac{\lambda}{32\eta^2} \left( \lambda^2 - 8(\eta^2 + f) \right), \quad (\text{C.3})$$

$$c_3(\lambda) = \frac{1}{8} \left( \lambda^2 - 8\eta^2 - 4f \right), \quad (\text{C.4})$$

$$c_4(\lambda) = \frac{\lambda}{2}, \quad (\text{C.5})$$

where

$$f = -\frac{1}{5} + 2\eta^2 \left( h + \frac{1}{5} \right), \quad (\text{C.6})$$

and  $\eta$  is given in (2.8). For  $h = h_{2211}$  one can verify that the two states (C.1), with  $\lambda$  given by

$$\lambda_{\pm} = \pm 2(1 + 2h) \sqrt{\frac{2}{(3 - h)(1 + 5h)}}, \quad (\text{C.7})$$

are  $\mathcal{W}_3$  primaries. Indeed, the states  $\Psi^{\pm} \equiv \Psi(\lambda_{\pm})$ , are eigenvalues of  $W_0$  and obey  $L_{+1}\Psi^{\pm} = L_{+2}\Psi^{\pm} = 0$ . We therefore have the two null-state conditions,

$$\Psi_+ = 0 \quad \Psi_- = 0, \quad (\text{C.8})$$

which give the relations (3.20) and (3.21).

## D Structure constants from the $\mathfrak{sl}_3$ Coulomb gas

We review the computation of the structure constants (2.41) that enter in the correlation function (3.1) and that can be computed through Coulomb gas, as done in [4].

### D.1 The first node

The first node of the diagram (3.1) corresponds to

$$\begin{array}{c}
 \vec{\alpha}_{2111} \\
 \diagdown \\
 \vec{\alpha}_M = \vec{\alpha}_R - b\vec{h}_i \\
 \diagup \\
 \vec{\alpha}_R
 \end{array}
 \quad (D.1)$$

where  $\vec{h}_i$ ,  $i = 1, 2, 3$  are defined in (2.28). The structure constant  $C_{-b\vec{\omega}_1, \vec{\alpha}_R}^{\vec{\alpha}_R - b\vec{h}_i}$  is defined through the Coulomb gas three point function:

$$C_{-b\vec{\omega}_1, \vec{\alpha}_R}^{\vec{\alpha}_M} = \langle V_{2\vec{\alpha}_0 - \vec{\alpha}_M}(\infty) V_{-b\vec{\omega}_1}(1) V_{\vec{\alpha}_R}(0) \rangle. \quad (D.2)$$

From the neutrality condition (2.49), the Coulomb gas representation of  $C_{-b\vec{\omega}_1, \vec{\alpha}_R}^{\vec{\alpha}_R - b\vec{h}_i}$  gives the following representation:

$$\begin{aligned}
 C_{-b\vec{\omega}_1, \vec{\alpha}_R}^{\vec{\alpha}_R - b\vec{h}_1} &= 1 \quad (n_1 = n_2 = m_1 = m_2 = 0) \\
 C_{-b\vec{\omega}_1, \vec{\alpha}_R}^{\vec{\alpha}_R - b\vec{h}_2} &= \int d^2t |t|^{2b^2} |t-1|^{2x} \quad (n_1 = 1, n_2 = m_1 = m_2 = 0) \\
 C_{-b\vec{\omega}_1, \vec{\alpha}_R}^{\vec{\alpha}_R - b\vec{h}_3} &= \int d^2t_1 \int d^2t_2 |t_1|^{2b^2} |t_1-1|^{2x} |t_2|^0 |t_2-1|^{2y} |t_1-t_2|^{2b^2} \\
 &\quad (n_1 = 1, n_2 = 1, m_1 = m_2 = 0)
 \end{aligned}
 \quad (D.3)$$

where  $x$  and  $y$  are defined in (5.13). Using the formula (see for instance Appendix 2 of chapter 7 in [14]):

$$\int d^2t |t|^{2a} |t-1|^{2b} = \pi \frac{\gamma(1+a)\gamma(1+b)}{2+a+b} \quad (D.4)$$

one derives:

$$\begin{aligned}
 C_{-b\vec{\omega}_1} &= 1 \\
 C_{-b\vec{\omega}_1, \vec{\alpha}_R}^{\vec{\alpha}_R - b\vec{h}_2} &= \pi \frac{\gamma(1+b^2)\gamma(1+x)}{\gamma(2+b^2+x)} \\
 C_{-b\vec{\omega}_1, \vec{\alpha}_R}^{\vec{\alpha}_R - b\vec{h}_3} &= \pi^2 \frac{\gamma(1+b^2)^2\gamma(1+y)\gamma(2+b^2+x+y)}{\gamma(2+b^2+y)\gamma(3+2b^2+y+x)}
 \end{aligned}
 \quad (D.5)$$

The above results coincide with those of Equation (1.51) in [4].

## D.2 The second node

The second node of the (3.1) is illustrated here:

$$\begin{array}{c}
 \vec{\alpha}_M = \vec{\alpha}_R - b\vec{h}_i \\
 \swarrow \quad \searrow \\
 \vec{\alpha}_{2211} = -b\vec{\rho} \\
 \vec{\alpha}_L = \vec{\alpha}_R - b\vec{\omega}_1
 \end{array} \tag{D.6}$$

We are interested here in the structure constants

$$C_{-b\vec{\rho}, \vec{\alpha}_M}^{\vec{\alpha}_L} = \langle V_{2\vec{\alpha}_0 - \vec{\alpha}_L}(\infty) V_{-b\vec{\rho}}(1) V_{\vec{\alpha}_M}(0) \rangle. \tag{D.7}$$

From the neutrality condition (2.49) and using formula (D.4), one obtains,

$$\begin{aligned}
 C_{-b\vec{\rho}, \vec{\alpha}_L + b\vec{\rho}}^{\vec{\alpha}_L} &= 1, \\
 C_{-b\vec{\rho}, \vec{\alpha}_L + b\vec{e}_1}^{\vec{\alpha}_L} &= \pi \frac{\gamma(1+b^2)\gamma(1+y+b^2)}{\gamma(2+y+2b^2)}
 \end{aligned} \tag{D.8}$$

The results coincides with Eq 1.56 of [4]<sup>5</sup>. More complicated is the case when  $\alpha_M = \alpha_L$ , related to the presence of multiplicities. The neutrality condition (2.49) is satisfied with one screening of type  $e_1$  and one screening of type  $e_2$ . One has

$$C_{-b\vec{\rho}, \vec{\alpha}_L}^{\vec{\alpha}_L} = \int \left( d^2 t_1 \right) \int \left( d^2 t_2 \right) |t_1|^{2b^2} |t_2|^{2b^2} |t_1 - 1|^{2x+2b^2} |t_2 - 1|^{2y} |t_1 - t_2|^{2b^2} \tag{D.9}$$

In the case  $y = x + b^2$ , the above integral is computed in Equation B.9 of [37] and one obtains

$$C_{-b\vec{\rho}, \vec{\alpha}_L}^{\vec{\alpha}_L} = 2\pi^2 \frac{\gamma(b^2)}{\gamma(b^2/2)} \prod_{i=0,1} \frac{\gamma(1 + (2+i)b^2/2)\gamma(1+y+ib^2/2)}{\gamma(2 + (3+i)b^2/2+y)} \quad \text{for } x + b^2 = y \tag{D.10}$$

In the most general situation, where  $x + b^2 \neq y$ , the above integral can be computed using the procedure explained in the Appendix A of [38]. The integral is then expressed as a quadratic combination of  ${}_3F_2$  hypergeometric functions computed at  $z = 1$ . We define a vector  $J = (J_1, J_2)$  where

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<sup>5</sup> In Eq 1.56 of [4] we point out a typos: the second product on the r.h.s runs from  $i + 1$  to  $n$  and not to  $n - 1$

$$J_1 = N_1 {}_3F_2 \left( -y, 2 + 3b^2, 1 + b^2; 3 + 4b^2 + x, 2 + 2b^2 | 1 \right) \quad (\text{D.11})$$

$$J_2 = N_2 {}_3F_2 \left( -b^2, 2 + x + y + 2b^2, 1 + x + b^2; 3 + 3b^2 + x + y, 2 + b^2 + y | 1 \right), \quad (\text{D.12})$$

with

$$\begin{aligned} N_1 &= \frac{\Gamma(2 + 3b^2)\Gamma(1 + x + b^2)\Gamma(1 + b^2)\Gamma(1 + b^2)}{\Gamma(3 + 4b^2 + x)\Gamma(2 + 2b^2)}, \\ N_2 &= \frac{\Gamma(2 + 2b^2 + x + y)\Gamma(1 + b^2)\Gamma(1 + y)\Gamma(1 + b^2)}{\Gamma(3 + 3b^2 + x + y)\Gamma(2 + b^2 + y)} \end{aligned} \quad (\text{D.13})$$

Using the notation  $[x] = \sin(\pi x)$ , we introduce the  $2 \times 2$  matrices

$$M_1 = \begin{pmatrix} [3b^2] & [x + 2b^2] \\ [y + b^2] & [y + 2b^2] \end{pmatrix}, \quad (\text{D.14})$$

$$M_2 = \begin{pmatrix} \frac{[b^2]^2}{[y+2b^2]} & \frac{[b^2][2b^2]}{[y+2b^2]} \\ \frac{[b^2][2b^2]}{[x+3b^2]} & \frac{[b^2]^2}{[x+3b^2]} \end{pmatrix}, \quad (\text{D.15})$$

and  $M = M_1^{-1}M_2$ . We obtained the following expression for the structure constant,

$$\begin{aligned} C_{-b\vec{\rho}, \vec{\alpha}_L}^{\vec{\alpha}_L} &= [b^2] [x + b^2] J_T M J \\ &+ \sum_{j=1,2} \left( [x + b^2] [y + b^2] J_1 M_{2j} J_j + [x + 2b^2] [y] J_2 M_{1j} J_j \right) \end{aligned} \quad (\text{D.16})$$

We have verified that the (D.16) coincides with (D.10) when  $x + b^2 = y$ .

## E A $\mathcal{W}_3$ basis

*We outline an algorithm to expand any  $\mathcal{W}_3$  state in terms of the basis states. The same operations allow us to compute the matrix elements in Section 3, following [6]*

### E.1 Definitions

We start with a number of simple definitions, all of which are self-evident, but we include them for completeness.

**Modes and products.** We refer to  $L_m$  as an  $L$ -mode, and to  $W_n$  as an  $W$ -modes. The indices  $m$  and  $n \in \mathbb{Z}$  are mode-numbers. We refer to a product of  $L$ -modes only as an  $L$ -product, a product of  $W$ -modes only as a  $W$ -product, and to a product of  $L$ -modes and

$W$ -modes as an  $LW$ -product. When an  $LW$ -product consists of one or more sequences of consecutive  $L$ -modes, followed by sequences of consecutive  $W$ -modes, *etc.*, we refer to each of these sequences as an  $L$ -sub-product, and  $W$ -sub-product.

**Normal order.** An  $L$ -product  $L_{m_1} \cdots L_{m_L}$  is normal-ordered if

$$m_1 \leq m_2 \leq \cdots \leq m_L \quad (\text{E.1})$$

that is, the mode-numbers increase from left to right. Similarly, a  $W$ -product  $W_{n_1} \cdots W_{n_W}$  is normal-ordered if

$$n_1 \leq n_2 \leq \cdots \leq n_W \quad (\text{E.2})$$

An  $LW$ -product is normal-ordered if all  $W$ -modes act first from the left on the highest-weight state, in normal-order, then all  $L$ -modes act second, also in normal-order.

**Inversion numbers.** An inversion number of an arbitrary product is the number of inversions, or permutations of nearest-neighbouring modes, that are required to put a product in a specific form. To quantify the degree of disorder of an arbitrary  $LW$ -product, we use four inversion numbers,

1.  $I_{LW+}$  is the inversion number of non-negative  $L$ -modes with respect to the  $W$ -modes. For each non-negative  $L$ -mode in an  $LW$ -product, we record the number of  $W$ -modes on its right. The sum of these numbers is  $I_{LW+}$

2.  $I_{LW-}$  is the inversion number of negative  $L$ -modes with respect to the  $W$ -modes. For each negative  $L$ -mode in an  $LW$ -product, we record the number of  $W$ -modes on its left. The sum of these numbers is  $I_{LW-}$

3.  $I_{WW}$  is the inversion number of  $W$ -modes with respect to each other. For each  $W$ -mode in an  $LW$ -product, we record the number of lower  $W$ -modes on its right. The sum of these numbers is  $I_{WW}$

4.  $I_{LL}$  is the inversion number of  $L$ -modes with respect to each other. For each  $L$ -mode in an  $LW$ -product, we record the number of lower  $L$ -modes on its right. The sum of these numbers is  $I_{LL}$

**Basis states.** The Hilbert space of a  $\mathcal{W}_3$  conformal field theory is spanned by a basis, the elements of which are created by the action of the negative  $W$ -modes in normal-order, followed by the action of the negative  $L$ -modes, also in normal-order,

$$|Y_L, Y_W, h, w\rangle = L_{m_1} \cdots L_{m_L} W_{n_1} \cdots W_{n_W} |h, w\rangle, \quad m_1 \leq \cdots \leq m_L < 0, \quad n_1 \leq \cdots \leq n_W < 0 \quad (\text{E.3})$$

where  $m_i < 0$ ,  $i = 1, \dots, L$ ,  $n_j < 0$ ,  $j = 1, \dots, W$ ,  $Y_L$  is a Young diagram with parts  $|m_L| \leq \cdots \leq |m_1|$ , and  $Y_W$  is a Young diagram with parts  $|n_W| \leq \cdots \leq |n_1|$ . The quantum



numbers  $h$  and  $w$  label the highest-weight state of the  $\mathcal{W}_3$  highest-weight representation that the state  $|Y_L, Y_W, h, w\rangle$  belongs to.

**Disordered states.** Any state that is *not* in the form (E.3) is a disordered state. We encounter disordered states in intermediate steps of computations, including those of matrix elements of descendant states, as in Section 3. The  $L$ -modes and  $W$ -modes in disordered states can be non-negative.

## E.2 Expanding a disordered state in terms of basis states

Following [1], any disordered state can be expressed as a linear combination of basis states. In Virasoro conformal field theories, the commutation relations are relatively simple, and can be used in a straightforward way to expand any disordered state in terms of the basis states. In  $\mathcal{W}_N$  conformal field theories, the commutation relations are more involved, which is related to the fact that  $\mathcal{W}_N$ ,  $N = 3, 4, \dots$ , is not a Lie algebra. It is instructive to see how one can expand any disordered state in terms of basis states, in the case of  $\mathcal{W}_3$  theories. We outline one way to do this, based on a systematic application of the  $\mathcal{W}_3$  commutation relations in Section 2.1. For the purposes of the algorithm that we outline in this appendix, it is convenient to re-write the commutation relations in a simple form that discards the coefficients.

$[L, W]$ , the  $L$ -mode is non-negative.

$$L_m W_n \sim W_n L_m + W_{m+n}, \quad 0 \leq m, \quad (\text{E.4})$$

that is, commuting a non-negative  $L$ -mode, from the left to the right of  $W_n$ , for any  $n \in \mathbb{Z}$ , we end up with two terms. We use this commutation to move a non-negative  $L$ -mode, from the left to the right of a  $W$ -mode. The first term has a lower  $I_{LW+}$  inversion number than the term on the left. The second term can be interpreted as the absorption of  $L_m$  into  $W_n$  to produce  $W_{m+n}$ , which leads to a shorter product of modes, the normal-ordering of which is a simpler problem than the one that we started with, thus the second term is also an improvement on the term on the left.

$[W, L]$ , the  $L$ -mode is negative.

$$W_n L_m \sim L_m W_n + W_{m+n}, \quad m < 0, \quad (\text{E.5})$$

that is, commuting a negative  $L$ -mode to the left of  $W_n$ ,  $n \in \mathbb{Z}$ , we end up with two terms. Each term on the right is an improvement on the term on the left for the same reasons as in section E.2.

### The commutator of two $W$ -modes.

$$W_n W_m \sim W_m W_n + L_{m+n} + \delta_{m+n,0} + \sum_{k=-\infty}^{\infty} : L_{-k} L_{m+n+k} :, \quad m < n \quad (\text{E.6})$$

that is, commuting a higher-mode  $W_n$  to the right of a lower-mode  $W_m$ , we end up with four terms. Each of the first three terms on the right is an improvement on the term on the left for the same reason as in section E.2. The fourth term is an improvement in the sense that it replaces two  $W$ -modes by two  $L$ -modes. The resulting state is easier to normal-order than the one that we started with, since  $L$ -modes obey simpler commutation relations than the  $W$ -modes. In fact, the current  $L$ -modes are normal-ordered with respect to each other.

Following an application of (E.6), the initial state is replaced by a sum over infinitely-many states. To be able to expand a disordered state in terms of a linear combination of finitely-many basis states, in finitely-many number of steps, we must make sure that the sum, produced by (E.6) is finite. While the number of states produced in (E.6) is formally infinite, only finitely-many products survives. This is because, except in finitely-many cases, all new states contain a mode that is sufficiently-positive to kill the state it is in. This is because the number of  $W$ -modes to the right of the new pair of  $L$ -modes is finite, and the degree of each of these modes, whether positive or negative, is also finite. Using (E.4), a sufficiently-positive  $L$ -mode can be moved to the right, producing either a sufficiently-positive  $L$ -mode or a sufficiently-positive  $W$ -mode that kills the highest-weight state.

While the number of states increases, each descendant state is an improvement over its ancestor. Because the initial degree of disorder is finite, the increment in the number of states is finite, and the disorder decreases by a finite amount at each step, the algorithm will terminate in finitely-many steps.

### The commutator of two $L$ modes.

$$L_n L_m \sim L_m L_n + L_{m+n} + \delta_{m+n,0}, \quad m < n \quad (\text{E.7})$$

that is, commuting the higher-mode  $L_n$  to the right of the lower-mode  $L_m$ , we end up with two terms. Each of these terms is an improvement on the term on the left for the same reason as in section E.2.

**Finitely-many steps.** We consider finite-level  $\mathcal{W}_3$  states, constructed by the action of finitely-many  $L$ -modes and finitely-many  $W$ -modes, all with finite mode-numbers. We emphasise finiteness because the algorithm that we propose is iterative. For an iterative algorithm to make sense, it must terminate after a finitely-many steps. To show that the algorithm that we propose is finite, we use the inversion numbers, defined above, to measure how far we are from our goal. In this appendix, our goal is to expand an arbitrary disordered  $\mathcal{W}_3$  state in terms of basis  $\mathcal{W}_3$  states.

**The algorithm.** We outline an iterative algorithm. Each iteration takes an  $LW$ -product with a finite degree of disorder as an input, and generates finitely-many descendant  $LW$ -products as outputs. Each output  $LW$ -string has **1.** a lower degree of disorder, **2.** a smaller number of  $L$ -modes, or **3.** a smaller number of  $W$ -modes, than the input  $LW$ -product. After a finite number of steps, every output  $LW$ -product is normal-ordered, and the algorithm terminates.

**Four steps.** The algorithm is based on the iteration of four operations, based on the four simplified commutators (E.4), (E.5), (E.6), and (E.7).

**Step 1.** We scan the input  $LW$ -product from right to left, and locate the first non-negative  $L$ -mode, which has a negative  $L$ -mode or any  $W$ -mode to its right. We use (E.4) to move this mode one step to the right. If the mode to the right is a negative  $L$ -mode, we use (E.7) to move the non-negative  $L$ -mode to the right again. Repeating step 1 finitely-many times, the result is a linear combination of  $LW$ -products that has no non-negative  $L$ -modes.

**Step 2.** We scan the state from left to right, we locate the first negative  $L$ -mode, which has any  $W$ -mode to its left. We use (E.5) to move this mode one step to the left. Repeating step 2 finitely-many times, the result is a linear combination of  $LW$ -products. Each of these products consists of a generally disordered  $L$ -product, with negative  $L$ -modes only, on the left, and a generally disordered  $W$ -product on the right.

**Step 3.** Consider each of the  $LW$ -products obtained at the end of the final iteration of step 2 above. Scan the  $LW$ -product from right to left, and locate the first positive  $W$ -mode, which has a negative  $W$ -mode to its right. Use (E.6) to move this mode one step to the right.

**Repeating steps 1 and 2.** One of the descendants produced in step 3 is a two-operator  $L$ -product in the middle of a previously-pure  $W$ -product. To clear this, we need to repeat step 1 and step 2 again. Iterating steps 1 and 2 finitely-many times, we end up with  $LW$ -products, such that each of these products consists of a disordered  $L$ -product to the left, and a normal-ordered  $W$ -product to the right.

**Step 4.** We consider each  $LW$ -product produced in the final iteration of step 3, and use (E.7) to order the  $L$ -sub-product. We end up with a set of  $LW$ -products such that the  $L$ -product is normal-ordered and the  $W$ -product is normal-ordered. This concludes the algorithm.

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